1 Review of Newton's Laws

1.1 Introduction

In this chapter Newton's laws of motion are reviewed, and some applications are presented that introduce you to some basic computational tools. In addition to the simple harmonic oscillator, the forced harmonic oscillator with damping and the free-falling body under the action of gravity with air resistance are also considered. While you should have been exposed to some of the basic physics in this chapter, the material is presented with the goal that we will ultimately need to incorporate the formulae into a computer.

Before embarking on a journey regarding the application of Newton's laws, the basic foundation of this text, let's delve into a brief biography of Newton. Indeed, before Newton, there were Galileo (1564–1642) and Huygens (1629–1697), on whose great experimental works dealing with the inertia of a body, motions of projectiles, and the oscillations of pendulums Newton built his laws.

Isaac Newton (1642–1727) was a physicist and a mathematician who was born in Woolsthorpe, a sheep farm town in Lincolnshire, England. He studied at Cambridge University. In around 1665 the fall of an apple is said to have suggested the train of thought that led to the universal law of gravitation, which is duly named after him, and which we study in this text in Chapter 9. On his own, he also studied properties of light, concluding that white light is a mixture of colors that can be separated by refraction. A popular telescope, known today as a Newtonian reflecting telescope, was originally devised by him. It is a telescope based on a reflecting parabolic mirror rather than a refracting lens. He became professor of mathematics at Cambridge in 1669, where he resumed his work on gravitation, expounded finally in his famous *Philosophiae naturalis principia mathematica* (*Mathematical Principles of Natural Philosophy,* 1687). In 1696 he was appointed warden of the Mint, and was master of the Mint from 1699 until his death. He also sat in Parliament on two occasions, was elected President of the Royal Society in 1703, and was knighted in 1705. During his life he was involved in many controversies, notably with Leibniz over the question of priority in the discovery of calculus.

Newton laid the groundwork for many of his contributions in physics, mathematics, and astronomy at the age of 23. In his own words "I was in the prime of my age for invention, and minded mathematics and philosophy [science] more than any time since." He also said, "I do not know what I may appear to the world; but to myself I seem to have been only like a boy playing on the seashore, and diverting myself in now and then finding a smoother pebble or a prettier shell than ordinary, whilst the great ocean of truth lay all undiscovered before me." In his accomplishments, he seemed to have been guided by a simple principle: "Truth is ever to be found in the simplicity, and not in the multiplicity and confusion of things."

The French mathematician Joseph Louis Lagrange (1736–1813) had this to say: "Newton was the greatest genius who ever lived, and the most fortunate, for there cannot be more than once a system of the world to establish."

Newton's epitaph refers to a person who, "by vigor of mind almost divine, the motions and figures of the planets, the paths of comets, and the tides of the seas first demonstrated." Newton was buried at Westminster Abbey, where the inscription on his tomb reads: "Let Mortals rejoice that there has existed such and so great an ornament of the human race."

In this chapter, we aim only to write down Newton's laws for the simple harmonic oscillator, the forced harmonic oscillator with damping, and a falling body under the action of gravity with air resistance and to see how one can use the computer to solve the equations. While we could go ahead and obtain the analytic solutions to these popular problems, this task is postponed to Chapters 2 and 3. In this chapter we seek to obtain numerical solutions to these problems and explore them using the computational language of MATLAB. In Chapter 2 we seek to show that these numerical results do in fact agree with the analytic results obtained in that chapter.

1.2 Basic Ideas of Newton's Laws of Motion

Newton's laws deal with classical mechanics, i.e., the area of physics that deals with large bodies, as opposed to the atomic world. The motion of a body refers to the change in position of that body as a function of time. Thus, there are three laws of motion.

First Law

This is commonly known as the law of inertia. It can be stated as follows: a body at rest remains at rest, or in motion, if in motion, in a straight line, unless it is applied an external net force. This can be stated mathematically as

$$\mathbf{F}_{net} = \sum_i \mathbf{F} = 0, \tag{1.2.1}$$

where, $\mathbf{F}_i$ is the ith external force. The total sum of all the present external forces results into the net force $\mathbf{F}_{net}$. This net force must equal zero for the first law to apply. When the first law holds for a given system, the system is said to form an *inertial frame of reference*. When a system accelerates, such system is said to form a *non-inertial frame of reference* and the first law no longer holds. In essence, the concept of acceleration enters through the second law of motion.

For example, consider the body shown in Figure 1.1a, which experiences no acceleration. If the forces $\mathbf{F}_1$ and $\mathbf{F}_2$ are known, then by Newton's first law we must have that $\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 = 0$. It follows that $\mathbf{F}_3 = -\mathbf{F}_1 - \mathbf{F}_2$.

Second Law

Figure 1.1(a) refers to a situation when the body is in equilibrium. It experiences no acceleration and therefore no net force. In the presence of an applied external net force, however, a body experiences an acceleration that is proportional to the net force and inversely proportional to the body's mass. Thus we write

$$\mathbf{a} = \mathbf{F}_{net}/m = \sum_i \mathbf{F}_i/m. \tag{1.2.2}$$

This is simply depicted in Figure 1.1(b). This also means that since $\mathbf{a} = d\mathbf{v}/dt$ then $m\mathbf{a} = d\mathbf{p}/dt$, where $\mathbf{p} = m\mathbf{v}$ is the linear momentum of the body moving with velocity $\mathbf{v}$. We will return to this shortly. For now, consider an example of the body shown in Figure 1.1(c), which experiences an acceleration in the positive x direction. While a full treatment of vectors will be made in Chapter 5, here we use the vector notation $\mathbf{F}_1 = (F_{1x}, F_{1y})$, $\mathbf{F}_2 = (F_{2x}, F_{2y})$ for the respective x and y components of the forces. Similarly, for the acceleration we write $\mathbf{a} = (a_x, a_y)$ and for the normal force we write $\mathbf{N} = (0, N_y)$. The weight of the object is $m\mathbf{g}$, due to the gravitational force, where $\mathbf{g} = (0, -g)$ whose direction is in the $-y$ direction. Because there is no acceleration in the y direction and if the forces $\mathbf{F}_1$ and $\mathbf{F}_2$ are known, then by Newton's first

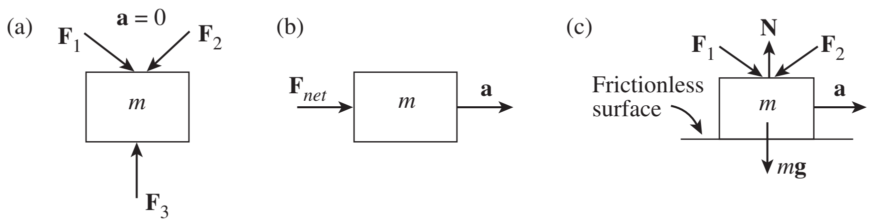

FIGURE 1.1

law it follows that $F_{1y} + F_{2y} + N - mg = 0$ and so the magnitude of the normal force is $N = N_y = -F_{1y} - F_{2y} + mg$. However, we use Newton's second law to obtain the acceleration; that is, $F_{1x} + F_{2x} = ma_x$, and so the magnitude of the acceleration is $a = a_x = (F_{1x} + F_{2x})/m$. In this example, the components of the forces in the x and y directions would have to be known to obtain the acceleration and the normal force respectively. Also, the sign or direction of the acceleration will be according to the sign or direction of the net force as defined by (1.2.2).

This law is responsible for our ability to predict the position and velocity of a body as a function of time, given appropriate initial conditions. An accelerated system cannot represent an inertial frame of reference because it does not obey Newton's first law. The concept of a net force is very important. For example, imagine a body to be the only body in the universe and at rest. There are no external forces on it, so the first law applies, and the body will remain at rest forever. If this body had an initial velocity associated with it, it would still represent an inertial frame of reference moving at the same velocity forever because there are no forces acting on it. The body in Figure 1.1(a) has forces acting on it. But because the sum of those forces equals a zero net force, the body, moving or not, does not accelerate and remains an inertial frame of reference. Its velocity will remain the same always. However, if one considers a different situation, one in which a body rests on a table on Earth, the body, while at rest, can also have forces acting upon it. In other words, a body can be at rest even if there are forces present. To understand that kind of situation better, one applies Newton's third law of motion.

Third Law

Simply stated, for every action there is an equal and opposite reaction. To state this law mathematically, imagine two bodies interacting in free space and which we label i and j, with respective masses m_i and m_j. Referring to Figure 1.2(a), if we let $\mathbf{f}_{ij}$ be the force exerted on body i due to body j, and $\mathbf{f}_{ji}$ be the force exerted on body j due to body i, then the third law says that

$$\mathbf{f}_{ij} = -\mathbf{f}_{ji}. \tag{1.2.3a}$$

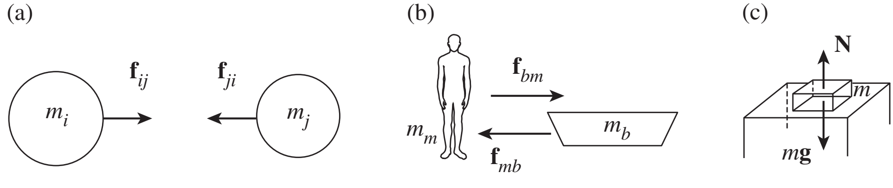

FIGURE 1.2

A useful application of this law is that, in conjunction with the second law, one can obtain information about the relationship between the acceleration of each body. The acceleration of body i is

$$\mathbf{a}_i = \mathbf{f}_{ij}/m_i, \tag{1.2.3b}$$

and using (1.2.3a) the acceleration of body j becomes

$$\mathbf{a}_j = \mathbf{f}_{ji}/m_j = -\mathbf{f}_{ij}/m_i. \tag{1.2.3c}$$

Now, because the force each body exerts on the other is equal in magnitude but opposite in direction as in Figure 1.2(a), we can express the acceleration of body j in terms of the acceleration of body i from (1.2.3c) as

$$\mathbf{a}_j = -\mathbf{a}_i m_i/m_j. \tag{1.2.3d}$$

From this expression it is readily seen that if the masses are equal, the accelerations are equal in magnitude but opposite in direction.

Further, consider Figure 1.2(b). Suppose a 50-kg (m_m) man, who is standing on a frictionless surface, pushes on a 20-kg (m_b) boat, which is resting on a water surface, with a force of $f_{bm} = 30$ N. Assuming there is no fluid viscosity, the boat experiences an acceleration of $a_b = f_{bm}/m_b = 30/20 = 1.5\ \mathrm{m/s^2}$. However, the boat exerts an equal and opposite force on the man ($f_{mb} = -f_{bm}$), so that the man experiences an acceleration of $a_m = -a_b m_b/m_m = -1.5(20/50) = -0.6\ \mathrm{m/s^2}$. Over time, the lighter mass boat will be displaced by a larger distance than the more massive man.

It is noted that when we apply Newton's second law of motion on a particular system, it is also implied that the first and third laws are automatically to be taken into account. For example, if we apply the second law to an object that is not accelerating, this is equivalent to solving Newton's first law because the acceleration of the system is zero. We say that such a system is a *system in mechanical equilibrium* or simply in equilibrium. An example of this is shown in Figure 1.2(c). A book resting on a table is in equilibrium. The normal force acting on the book is a force acting on the surface of the book, at its interface between it and the table and perpendicular to it. It is a reaction of the table on the book. In the special case when there are no other forces present other than the ones shown in the figure, because the book is not accelerating, it is possible to find its value. Here there are two forces only; one is the normal force, which points up $\mathbf{N} = (0, N)$, and the other is the gravitational force, which points down $m\mathbf{g} = (0, -mg)$. Adding these two forces we have $\mathbf{N} + m\mathbf{g} = 0$, or $\mathbf{N} = -m\mathbf{g}$. Thus they are equal in magnitude and opposite in direction. Similarly, when a system is accelerating due to a net force, the third law must be considered carefully when the net force is accounted for. For example, in the case when friction (kinetic) is present, it is the reaction from a surface on an object and it is always pointed in a direction that opposes the motion of the object.

If we express the net force in terms of momentum, Newton's first law applied to systems in equilibrium leads to momentum conservation. More specifically, writing the net force as

$$\mathbf{F} = m\frac{d\mathbf{v}}{dt} = \frac{d\mathbf{p}}{dt}, \tag{1.2.4}$$

where $\mathbf{p} = m\mathbf{v}$ is the body's linear momentum and where we have taken the body's mass to be a constant, then applying Newton's first law, $\mathbf{F} = 0$, the right of (1.2.4) means that

$$\mathbf{p} = \text{constant}, \tag{1.2.5}$$

or that *momentum is conserved.*

The preceding three laws, along with the time derivative relations associated with the general position $\mathbf{r} = (x, y, z)$, velocity $\mathbf{v} = (v_x, v_y, v_z)$, and acceleration $\mathbf{a} = (a_x, a_y, a_z)$ of a body

$$\mathbf{v} = \frac{d}{dt}\mathbf{r}, \tag{1.2.6a}$$

$$\mathbf{a} = \frac{d}{dt}\mathbf{v} = \frac{d^2}{dt^2}\mathbf{r}, \tag{1.2.6b}$$

allow us to obtain information about the object's position, velocity, and acceleration as a function of time. For example, consider the motion of a particle in one dimension with an acceleration $\mathbf{a} = a_x\hat{i}$ and where we subsequently replace a_x with a. Its velocity as a function of time can be obtained directly by integrating (1.2.6b)

$$\int dv = \int a\,dt. \tag{1.2.7a}$$

If we assume the acceleration is constant, the integration of (1.2.7a) leads to the one-dimensional velocity,

$$v(t) = v_0 + at, \tag{1.2.7b}$$

where v_0 is the initial value of the velocity in the x direction. The displacement as a function of time is obtained by integrating the one-dimensional form of (1.2.6a),

$$\int dx = \int v(t)\,dt, \tag{1.2.7c}$$

or

$$x(t) = x_0 + v_0 t + \frac{1}{2}at^2. \tag{1.2.7d}$$

If the acceleration is not constant, Equation (1.2.7a) can still be used to obtain $v(t)$ and the result can be used with Equation (1.2.7c) to obtain the corresponding $x(t)$. We will investigate this further in the next chapter. Sometimes, it is useful to be able to simulate the motion of a particle using a computational approach. The simplest way to do that is to use the Euler method. According to this method, suppose that we need the solution of a differential equation of the form

$$\frac{dy}{dx} = f(x, y) = y', \tag{1.2.8}$$

for $y(x)$ in the range $x_0 \le x \le x_f$. Then we first write this in the approximate form

$$y' \approx \frac{y(x+h) - y(x)}{h} = \frac{y_{i+1} - y_i}{h}, \tag{1.2.9}$$

where $h = (x_f - x_0)/N$, with N the number of desired steps in the interval $[x_0, x_f]$. Here $y_i \equiv y(x_i)$ and we suppose that for $i = 0$, $y_0 = y(x_0)$ is given. Thus combining (1.2.8) and (1.2.9) we see that

$$\frac{y_{i+1} - y_i}{h} = f(x_i, y_i), \tag{1.2.10}$$

which gives the recursion formula for the value of y as a function of x as

$$y_{i+1} = y_i + hf(x_i, y_i), \text{ with } x_{i+1} = x_i + h. \tag{1.2.11}$$

If we apply the Euler method to our one-dimensional time-dependent problem for which we have

$$\frac{dv}{dt} = a \quad \text{and} \quad \frac{dx}{dt} = v, \tag{1.2.12}$$

then we get the Euler approach relation for the velocity by replacing $y_i \to v_i$, $x_i \to t_i$, $f \to a$, and $h \to \Delta t$ in (1.2.11) to obtain

$$v_{i+1} = v_i + a_i \Delta t, \tag{1.2.13}$$

with the initial velocity, $v_{i=0} = v_0$. Similarly, this resulting velocity along with the second equation of (1.2.12) can in turn be used to obtain the position, but this time we make the substitutions $y_i \to x_i$, $x_i \to t_i$, $f \to v$, and $h \to \Delta t$ in (1.2.11), to write

$$x_{i+1} = x_i + v_{i+1} \Delta t. \tag{1.2.14}$$

with initial position, $x_{i=0} = x_0$, and with time evolving simply as

$$t_{i+1} = t_i + \Delta t, \tag{1.2.15}$$

where $t_{i=0} = t_0 \equiv 0$. The process stops when the desired number of N steps is reached.

The actual Euler form of (1.2.14) uses v_i instead of v_{i+1}, but the presence of v_{i+1} produces more accurate results because a more recent value of the velocity is used. This slight modification of the Euler method is known as the Euler–Cromer method, which is equivalent to a so-called Verlet algorithm. The approximation (1.2.14) works well because, in essence, it involves a correction to x_{i+1} to second order in Δt (see Problem 1.12) as opposed to terms up to first order in Δt if we were to use v_i instead of v_{i+1}. Several algorithms exist to solve differential equations numerically, but the Euler–Cromer method is the simplest method that yields accurate solutions for oscillatory problems. Also, keep in mind that decreasing the step size and increasing the number of steps, in general, yields more accurate results, but not always. When the number of steps used is too large and the step size is too small, the calculational time can become too long, and it's probably wise to seek an alternate algorithm.

1.3 Numerical Applications of Newton's Second Law Using the Modified Euler Method

In this section, Newton's laws of motion will be employed to carry out a numerical investigation of the motion of a mass under two different conditions. In the first case, a mass is attached to the end of a massless spring under the action of a damping as well as a driving force. In the second case, the mass is under the action of a constant gravitational force and a force due to air resistance. In this book, MATLAB will be

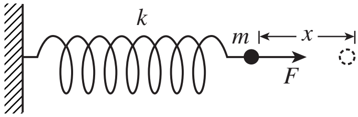

FIGURE 1.3

used throughout for numerical applications. An introduction to MATLAB is available in Appendix A.

Spring Mass System

In its simplest form, one can consider a mass attached to the end of a massless spring while the opposite end of the spring is held fixed, as shown in Figure 1.3.

The figure shows that applying a force F on the mass has the effect of displacing it by amount x. By Hooke's law, the magnitude of the applied force is proportional to the displacement x. In the absence of damping and driving forces, according to Newton's third law, the spring responds with a force equal in magnitude but opposite in direction,

$$F_s = -kx(t). \tag{1.3.1}$$

According to Newton's second law, the bob's acceleration is $a(t) = -kx(t)/m = d^2x/dt^2$, which is a second-order linear differential equation for the bob's position as a function of time. The analytic solution will be investigated in Chapter 3. For now it is convenient to investigate it numerically using the Euler–Cromer method of the previous section.

The numerical solution can be carried out using Equations (1.2.13–1.2.15). Suppose we let $k = 1000$ N/m, $x_0 = 0.1$ m, $v_0 = 0.0$, and we wish to obtain $x(t)$ in the time interval $[0, t_{\max}]$, with $t_{\max} = 1$ s, and $N = 10$ steps, so that $\Delta t = t_{\max}/N = 0.1$ s. Thus, specifically, our numerical equations for a mass of 5 kg become

$$a_i = -1000x_i/5, \quad v_{i+1} = v_i + 0.1a_i, \quad \text{and} \quad x_{i+1} = x_i + 0.1v_{i+1}. \tag{1.3.2a}$$

Table 1.1 has the results for the 10 steps. For the initial values we have,

$$t_0 \equiv 0, \quad v_0 \equiv 0, \quad x_0 \equiv 0.1, \quad a_0 = -1000(0.1)/5 = -200(0.1) = -20. \tag{1.3.2b}$$

Table 1.1 Results from the recursion formulas (1.2.15, 1.3.2a–c)

Initial Values: $t_0 \equiv 0, \Delta t = 0.1, v_0 \equiv 0, x_0 \equiv 0.1, a_0 = -20$					
i	$i+1$	$t_{i+1} = t_i + \Delta t$	$v_{i+1} = v_i + 0.1a_i$	$x_{i+1} = x_i + 0.1v_{i+1}$	$a_{i+1} = -200x_{i+1}$
0	1	0.1	–2.0	–0.1	20
1	2	0.2	0.0	–0.1	20
2	3	0.3	2.0	0.1	–20
3	4	0.4	0.0	0.1	–20
4	5	0.5	–2.0	–0.1	20
5	6	0.6	0.0	–0.1	20
6	7	0.7	2.0	0.1	–20
7	8	0.8	0.0	0.1	–20
8	9	0.9	–2.0	–0.1	20
9	10	1.0	0.0	–0.1	20

The next step, keeping $i = 0$ and using (1.3.2a and b), gives

$$t_1 = t_0 + 0.1 = 0.1, \quad v_1 = v_0 + 0.1(a_0) = 0 + 0.1(-20) = -2,$$
$$x_1 = x_0 + 0.1(v_1) = 0.1 + 0.1(-2) = -0.1, \tag{1.3.2c}$$
$$a_1 = -200(x_1) = -200(-0.1) = 20.$$

Table 1.1 shows the succeeding steps according to the formulas (1.2.15 and 1.3.2a). We notice that the position, velocity, and acceleration of the particle tend to be oscillatory, although not very smoothly so. We also notice that the sign of the acceleration is always opposite to that of the displacement. This is of course a consequence of any simple harmonic motion system, as expected. At this point, it is convenient to develop a MATLAB program that incorporates the above numerical algorithm and that can allow us to input the initial parameters of the model. By increasing the number of steps involved in the calculation, we can also increase the accuracy of the calculation. However, before developing such a program, we might as well include damping and driving forces to the model.

A standard way to model the damping experienced by a mass at the end of a spring is to notice that a resistive force tends to oppose the change in the motion; thus we write $F_R = -Cv$. If one were to attach a driving force on the mass at the end of the spring, it is common for such a driving force to be harmonic in nature. A possible form for such a force is $F_D = F_0 \sin(\omega t)$, where F_0 and ω are the amplitude and the frequency of the driving force. Including these two new forces to the original force of

Equation (1.3.1) associated with the spring results in the more general expression for the differential equation of a mass-spring system

$$\frac{d^2x}{dt^2} = [-kx(t) - Cv(t) + F_0 \sin(\omega t)]/m = a. \tag{1.3.3}$$

While the analytic solution of this equation will be tackled in Chapter 3, for now we continue with the numerical solution approach. We notice that depending on the values of the parameters such as k, C, F_0, and ω, a wide range of behaviors can be investigated. The MATLAB program, `ho1.m`, that is capable of investigating such motion follows. Recall from Appendix A that the lines beginning with "%" are comment lines. Some of these comments explain what the script does, and some give suggested values to use. Sample runs of this program are also shown.

SCRIPT

```
%ho1.m
%Calculation of position, velocity, and acceleration for a harmonic
%oscillator versus time. The equations of motion are used for small time intervals
clear;
%NPTS=100;TMAX=1.0;%example Maximum number of points and maximum time
TTL=input(' Enter the title name TTL:','s');%string input
NPTS=input(' Enter the number calculation steps desired NPTS: ');
TMAX=input(' Enter the run time TMAX: ');
NT=NPTS/10;%to print only every NT steps
%K=1000;M=5.0;C=0.0;E=0.0;W=0.0;x0=0.1;v0=0.0;% example Parameters
K=input(' Enter the Spring constant K: ');
M=input(' Enter the bob mass M: ');
C=input(' Enter the damping coefficient C: ');
E=input(' Enter the magnitude of the driving force E: ');
W=input(' Enter the driving force frequency W: ');
x0=input(' Enter the initial position x0: ');% Initial Conditions
v0=input(' Enter the initial velocity v0: ');% Initial Conditions
t0=0.0;% start at time t=0
dt=TMAX/NPTS;%time step size
fprintf(' Time step used dt=TMAX/NPTS=%7.4f\n',dt);%the time step being used
F=-K*x0-C*v0+E*sin(W*t0); % initial force
a0=F/M;% initial acceleration
fprintf('    t       x       v       a\n');%output column labels
v(1)=v0;
x(1)=x0;
a(1)=a0;
t(1)=t0;
fprintf('%7.4f %7.4f %7.4f %7.4f\n',t(1),x(1),v(1),a(1));%print initial values
```

```
for i=1:NPTS
    v(i+1)=v(i)+a(i)*dt;                     %new velocity
    x(i+1)=x(i)+v(i+1)*dt;                   %new position
    t(i+1)=t(i)+dt;                          %new time
    F=-K*x(i+1)-C*v(i+1)+E*sin(W*t(i+1));  %new force
    a(i+1)=F/M;                              %new acceleration
% print only every NT steps
    if(mod(i,NT)==0)
        fprintf('%7.4f %7.4f %7.4f %7.4f\n',t(i+1),x(i+1),v(i+1),a(i+1));
    end;
end;
subplot(3,1,1)
plot(t,x,'k-');
ylabel('x(t) (m)','FontSize',14);
h=legend('position vs time'); set(h,'FontSize',14);
title(TTL,'FontSize',14);
subplot(3,1,2)
plot(t,v,'b-');
ylabel('v(t) (m/s)','FontSize',14);
h=legend('velocity vs time'); set(h,'FontSize',14)
subplot(3,1,3)
plot(t,a,'r-');
ylabel('a(t) (m/s^2)','FontSize',14);
xlabel('time (sec)','FontSize',14);
h=legend('acceleration vs time'); set(h,'FontSize',14)
```

We have performed a first rough run of `ho1.m` in order to reproduce the results of the simple harmonic oscillator of Table 1.1. While generally it is suggested to pick a large number of steps, the program's initial run that follows performs NPTS (=10) steps in the calculation. Should the value of NPTS be increased, the calculation is more accurate, but only 10 sets of instantaneous values are printed by the script as it is designed. However, the plots produced contain all the calculated results. The input/output of the first run follows and can be found in the file `ho1_1.txt`, and is followed by the resulting MATLAB plot of Figure 1.4.

OUTPUT

```
ho1_1.txt
>> ho1
 Enter the title name TTL:Simple Harmonic Oscillator - rough
 Enter the number calculation steps desired NPTS: 10
 Enter the run time TMAX: 1
 Enter the Spring contant K: 1000
```

```
Enter the bob mass M: 5
Enter the damping coefficient C: 0
Enter the magnitude of the driving force E: 0
Enter the driving force frequency W: 0
Enter the initial position x0: 0.1
Enter the initial velocity v0: 0
Time step used dt=TMAX/NPTS= 0.1000
   t       x       v       a
0.0000  0.1000  0.0000 -20.0000
0.1000 -0.1000 -2.0000  20.0000
0.2000 -0.1000  0.0000  20.0000
0.3000  0.1000  2.0000 -20.0000
0.4000  0.1000  0.0000 -20.0000
0.5000 -0.1000 -2.0000  20.0000
0.6000 -0.1000  0.0000  20.0000
0.7000  0.1000  2.0000 -20.0000
0.8000  0.1000  0.0000 -20.0000
0.9000 -0.1000 -2.0000  20.0000
1.0000 -0.1000  0.0000  20.0000
```

The rough results shown in the `ho1_1.txt` file (reproduced here) agree with those calculated before in Table 1.1 and show how the program works.

Figure 1.4 shows that the acceleration is proportional to the negative of the displacement, as mentioned before. Also, from Table 1.1, we see that the velocity's maximum is

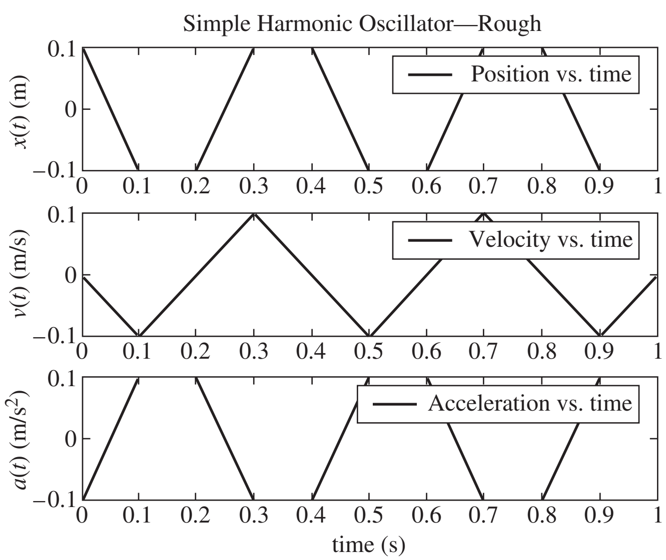

FIGURE 1.4 Rough calculation of position, velocity, and acceleration.

always shifted from the maximum of the displacement, similar to the behavior of the sine and cosine functions.

It is interesting to investigate the model further. The following are three sets of runs made with the program `ho1.m` that use more points for better accuracy. These are labeled as follows with the parameters used enclosed in parentheses:

Figure 1.5(a), "Simple Harmonic Oscillator" (NPTS=100, TMAX=10, K=1, M=1, C=0, E=0, W=0, x0=1, v0=0);

Figure 1.5(b), "Damped Harmonic Oscillator" (NPTS=200, TMAX=20, K=1, M=1, C=0.5, E=0, W=0, x0=1, v0=0);

Figure 1.5(c), "Forced Harmonic Oscillator with Damping" (NPTS=200, TMAX=20, K=1, M=1, C=0.5, E=0.1, W=0.8, x0=1, v0=0).

In the above runs, a higher value of NPTS is used when more accuracy is needed, which in turn decreases the step size, depending on the model. As a general rule, the more terms a model uses, the higher is the number of calculated points needed, along with a smaller step size, to minimize the error.

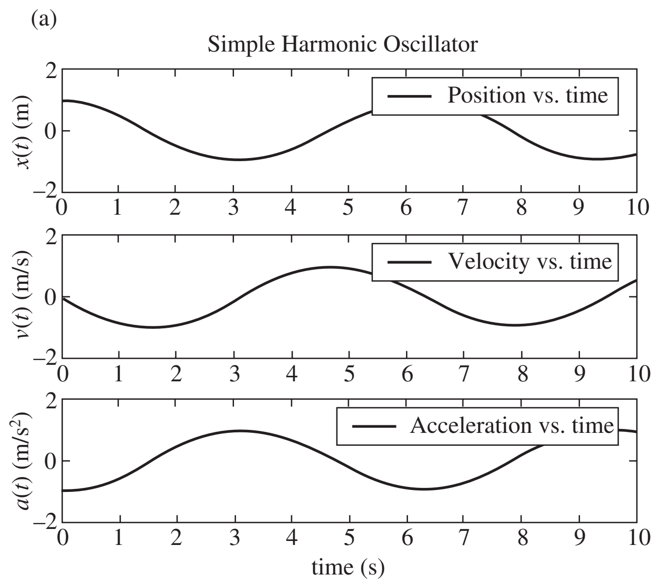

FIGURE 1.5 (a) "Simple Harmonic Oscillator" (NPTS=100, TMAX=10, K=1, M=1, C=0, E=0, W=0, x0=1, v0=0).

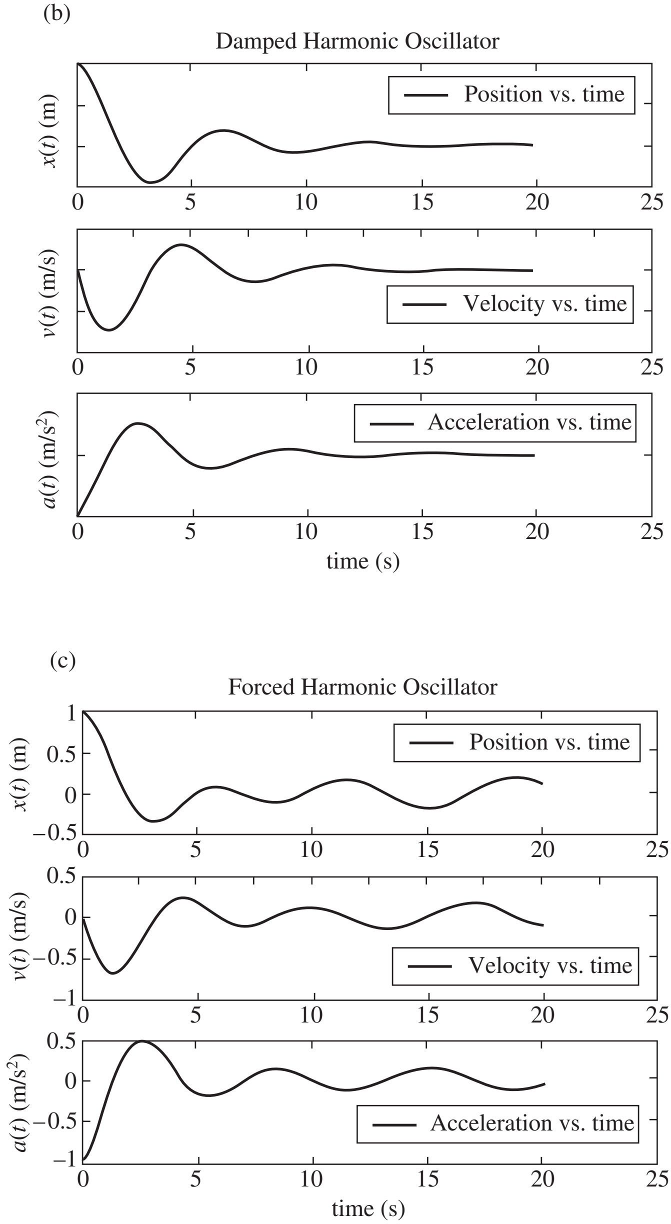

FIGURE 1.5 (b) "Damped Harmonic Oscillator" (NPTS=200, TMAX=20, K=1, M=1, C=0.5, E=0, W=0, x0=1, v0=0).
(c) "Forced Harmonic Oscillator with Damping" (NPTS=200, TMAX=20, K=1, M=1, C=0.5, E=0.1, W=0.8, x0=1, v0=0).

In Figure 1.5(a), we recognize the expected harmonic behavior obtained by increasing the number of points and using a smaller step value for dt. In Figure 1.5(b), the damped model results simply because the value of the damping coefficient is no longer zero. In Figure 1.5(c), the behavior is more complicated because the system tends to damp during a short time but after that, the system tends to respond to the driving force much more efficiently. Notice that the driving frequency is close to the value of 1, which for the present parameters corresponds to the natural frequency of the spring ($\sqrt{k/m}$). We will discuss this behavior later in Chapter 3 when we study the analytic solution.

Motion With and Without Air Resistance Under Constant Acceleration Due To Gravity

In the absence of air resistance and under a constant gravitational acceleration it is convenient to let $\mathbf{F} = -mg\,\hat{j} = m\mathbf{a}$, where $\hat{j}$ is a unit vector in the y-direction. Because only the y coordinate's motion is important we write

$$\frac{d^2y}{dt^2} = -g, \tag{1.3.4a}$$

where g is the constant value due to gravity. Because there is no air resistance, this equation refers to the motion of an object in *free fall*. The value of g is $9.80\,\mathrm{m/s^2}$ on Earth and $1.63\,\mathrm{m/s^2}$ on the moon. Integrating the preceding equation once over time gives the velocity, and integrating once more gives the displacement versus time

$$v = v_0 - gt, \quad \text{and} \quad y = y_0 + v_0 t - \frac{1}{2}gt^2, \tag{1.3.4b}$$

with the initial conditions v_0 for v, and y_0 for y. The inclusion of the effect due to air resistance on the motion is analytically more involved and is left for the next chapter. For now, a MATLAB program will be developed to include the resistive force due to air resistance. As Figure 1.6 shows, when the body is falling, the resistive force R points upward in a direction opposite to that of the velocity.

The net force associated with Figure 1.6 is written as

$$\mathbf{F}_{net} = [-mg + R(v)]\hat{j} = m\mathbf{a}, \tag{1.3.5}$$

where the force due to air resistance $R(v)$ is a function of velocity. In other words, this is a resistive force that is always pointed in a direction opposite to the direction of the motion, but its magnitude also depends on the magnitude of the velocity. We will

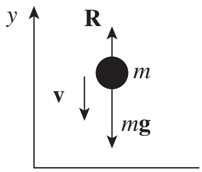

FIGURE 1.6

consider two popular models for the magnitude of R, one for which $R \propto -v$, and the other for which $R \propto -v^2$. A realistic use of a specific model for the drag term may very well depend on the actual value of the speed, the object's geometry, its size, as well as its mass, in addition to the properties of the substance through which the object moves. Thus we write R in two ways

$$R = -C_1 v, \quad \text{and} \quad R = -C_2 v^2. \tag{1.3.6}$$

In particular situations it is possible to have expressions for C_1 and C_2. For example, when a spherical object of radius a travels through a fluid of viscosity η, it is standard to take the linear model with $C_1 = 6\pi a\eta$, which is commonly referred to as Stokes' law. For a large object, like a weight falling through the air with an open parachute, one could employ the quadratic model with $C_2 = C_D \rho A/2$, where ρ is the air density, A is the object's cross-sectional area, and C_D is a drag coefficient in the range between 0.4 and 1.

Because the force due to gravity is constant, the velocity (or speed) of an object falling under gravity can achieve a high magnitude. There is a limit on how high such a speed can get, because referring to Figure 1.6, if the speed increases, then R will also increase. Therefore, there is a point where the net force will become zero, causing a zero acceleration and meaning that no further increase in speed is possible. In this case the speed reaches a terminal value v_t, commonly referred to as the *terminal velocity*. Thus in this limit from (1.3.5), we set

$$-mg + R(v_t) = 0. \tag{1.3.7}$$

Depending on the model for R used, two possible values of v_t can be found. Using (1.3.6–1.3.7), we can solve for the magnitude of the terminal velocity to get

$$v_{t_1} = mg/C_1, \quad \text{and} \quad v_{t_2} = \sqrt{mg/C_2}, \tag{1.3.8}$$

for the linear and the quadratic velocity models, respectively. Note that these are velocity magnitudes or speeds. Referring to Figure 1.6, the actual velocities are negative when objects are falling and positive when they are rising. For the quadratic model, the square of the speed is involved in the resistive force. In the analytic solution to the problem, in Chapter 2, the up and down motion is, therefore, treated separately. Here, the MATLAB program `ho2.m` solves this problem numerically. It accepts input from the user and produces results on a screen as well as the corresponding plots. The listing of the program follows. Thus, to incorporate the drag model of interest, we have included the "FLAG" input, which can take on the values of 0 or 1 depending on the drag model used, linear or quadratic, as mentioned previously.

SCRIPT

```
%ho2.m
%Calculation of position, velocity, and acceleration for a body in
%free fall with air resistance versus time.
%The equations of motion are used for small time intervals
clear;
%NPTS=200;TMAX=20.0;%example Maximum number of points and maximum time
TTL=input(' Enter the title name TTL:','s');%string input
NPTS=input(' Enter the number calculation steps desired NPTS: ');
TMAX=input(' Enter the run time TMAX: ');
NT=NPTS/10;%to print only every NT steps
%G=9.8;M=1.0;C=0.05;y0=0;v0=110;% example Parameters
G=input(' Enter value of gravity G: ');
M=input(' Enter the object mass M: ');
C=input(' Enter the drag coefficient C: ');
y0=input(' Enter the initial height y0: ');  % Initial Conditions
v0=input(' Enter the initial velocity v0: ');% Initial Conditions
FLAG=input(' Enter 0 (v drag) or 1 (v^2 drag) FLAG: ');
t0=0.0;% start at time t=0
dt=TMAX/NPTS;%time step size
if FLAG ==0
   F=-M*G-C*v0;           % initial force - case 1
   vt=abs(M*G/C);         % terminal velocity
   elseif FLAG==1
     F=-M*G-C*v0*abs(v0); % initial force - case 2
     vt=sqrt(M*G/C);      % terminal velocity
end;
%dt,FLAG, and vt used
fprintf(' FLAG=%1i, Time step dt=TMAX/NPTS=%5.2f, vt=%5.2f\n',FLAG,dt,vt);
a0=F/M;% initial acceleration
fprintf('    t       y       v       a\n');%output column labels
v(1)=v0;
```

```
y(1)=y0;
a(1)=a0;
t(1)=t0;
fprintf('%7.4f %7.4f %7.4f %7.4f\n',t(1),y(1),v(1),a(1));%print initial values
for i=1:NPTS
    v(i+1)=v(i)+a(i)*dt;                     %new velocity
    y(i+1)=y(i)+v(i+1)*dt;                   %new position
    t(i+1)=t(i)+dt;                          %new time
    if FLAG ==0
       F=-M*G-C*v(i+1);                      %new force - case 1
       elseif FLAG==1
         F=-M*G-C*v(i+1)*abs(v(i+1));        %new force - case 2
    end;
    a(i+1)=F/M;                              %new acceleration
% print only every NT steps
    if(mod(i,NT)==0)
        fprintf('%7.4f %7.4f %7.4f %7.4f\n',t(i+1),y(i+1),v(i+1),a(i+1));
    end;
end;
    plot(t,y,'k-',t,v,'b:',t,a,'r-.');
    ylabel('y (m), v (m/s), a (m/s^2)','FontSize',14);
    xlabel('time','FontSize',14);
    title(TTL,'FontSize',14);
    h=legend('position','velocity','acceleration',0); set(h,'FontSize',14)
```

An example run of this program, labeled "Falling with simple air resistance," including the input used, follows. In this run, we use the first drag model. This run produces the plot shown in Figure 1.7.

OUTPUT

```
ho2_1.txt
>> ho2
 Enter the title name TTL:Falling with simple air resistance
 Enter the number calculation steps desired NPTS: 200
 Enter the run time TMAX: 20
 Enter value of gravity G: 9.8
 Enter the object mass M: 1
 Enter the drag coefficient C: 0.05
 Enter the initial height y0: 0
 Enter the initial velocity v0: 110
 Enter 0 (v drag) or 1 (v^2 drag) FLAG: 0
 FLAG=0, Time step dt=TMAX/NPTS= 0.10, vt=196.00
```

```
   t        y        v        a
 0.0000   0.0000  110.0000 -15.3000
 2.0000 188.8649   80.8108 -13.8405
 4.0000 322.3215   54.4060 -12.5203
 6.0000 405.6549   30.5199 -11.3260
 8.0000 443.6466    8.9122 -10.2456
10.0000 440.6215 -10.6342 -9.2683
12.0000 400.4923 -28.3162 -8.3842
14.0000 326.7983 -44.3115 -7.5844
16.0000 222.7413 -58.7810 -6.8610
18.0000  91.2175 -71.8702 -6.2065
20.0000 -65.1530 -83.7109 -5.6145
```

In Figure 1.7, the acceleration tends to zero as the velocity begins to approach its terminal value, which for this particular set of parameters is 196 m/s.

Another run made using the same MATLAB program and labeled "Falling with v^2 air resistance," with the parameters: NPTS=200, TMAX=10, G=9.8, M=1, C=0.05, y0=0, v0=110, FLAG=1, is shown in Figure 1.8. The image was slightly magnified through MATLAB's figure viewer to show the curves better.

The calculated terminal velocity of Figure 1.8 is 14 m/s, a value much smaller than that of Figure 1.7. The difference is attributed to the stronger retardation effect

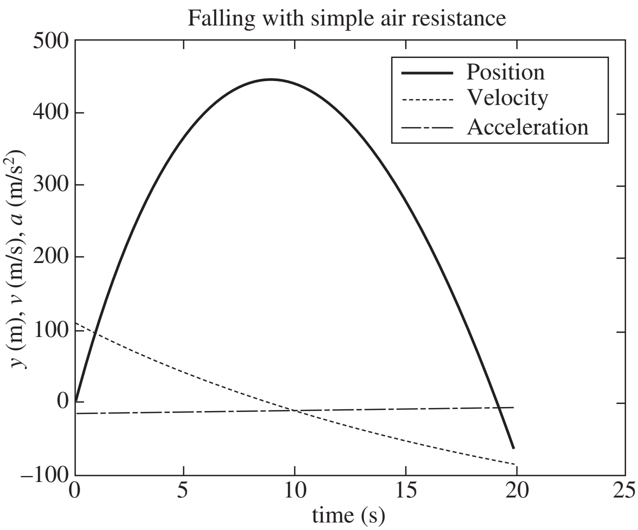

FIGURE 1.7 Falling Under Linear Air Resistance.

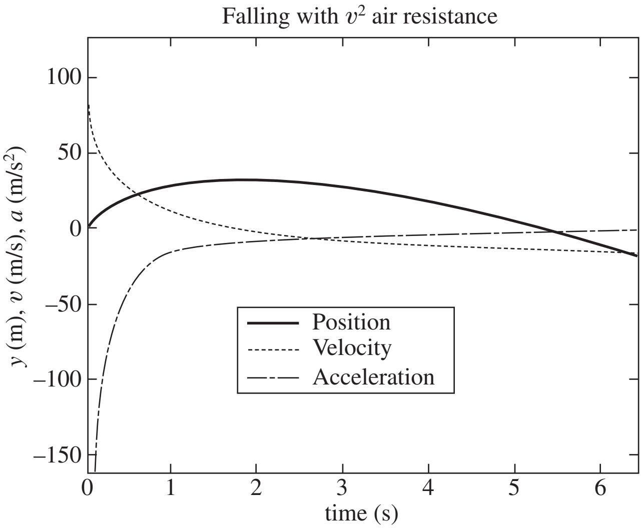

FIGURE 1.8 Falling under Quadratic Air Resistance.

associated with a drag force that's proportional to the square of the velocity. The acceleration decreases very rapidly to zero as well. Most of the parameters used in both Figures 1.7–1.8 are the same; the only difference in the runs is the value of the FLAG parameter, which determines the drag model used.

Chapter 1 Problems

1.1 Run the `ho1.m` script with $m = 1\,\text{kg}$, $\Delta t = 0.001$, $k = 1\,\text{N/m}$, and the rest of the parameters compatible with the simple harmonic oscillator. At $t = 0$, take $x_0 = 1\,\text{m}$, $v_0 = 0.0\ \text{m/s}$. From the output, find

- **a.** the period.
- **b.** the amplitude.
- **c.** Are the position and velocity related? How?

1.2 Use the program `ho1.m` along with the parameters $m = 1$ kg, $k = 1$ N/m, $x_0 = 1.5$ m, $v_0 = 0.0$ m/s, and a suitable value of Δt and do the following cases for the damped harmonic oscillator without a driving force: vary the value of C starting with 0.0 up to a value of 2.5 in steps of 0.5. Explain what happens to the period and amplitude as C varies.

1.3 Using a value of $C = 0.25$, for what value of frequency of the driving force does the highest amplitude occur in the damped harmonic oscillator? Use parameters similar to Problem 1.2 with a nonzero value of the driving force. What happens as C gets larger?

1.4 Use program `ho2.m` for free fall. Without air resistance obtain results for $x(t)$ and $v(t)$ and compare the results with the analytic expressions of Equations (1.3.4). Use the following initial values:

a. $y_0 = 0\,\text{m}$, $v_0 = 0.0\ \text{m/s}$,

b. $y_0 = 0\,\text{m}$, $v_0 = 100.0\ \text{m/s}$. Explain your results. Pick a reasonable value of mass, even though its value does not matter. Why?

1.5 A body falls with air resistance proportional to its velocity. A second body falls with air resistance proportional to the square of its velocity and a value of $C_2 = 0.01$. What value of C_1 should the first body have if it is to have the same terminal velocity as the second body? Run program `ho2.m` for both bodies and investigate the results for the two bodies with two sets of initial conditions:

a. $y_0 = 0\,\text{m}$, $v_0 = 0.0\ \text{m/s}$,

b. $y_0 = 0\,\text{m}$, $v_0 = 100.0\ \text{m/s}$. Assume $m = 1$ kg for both bodies and explain your results.

1.6 What are the units of the drag coefficients for the air resistance models that are linear and quadratic in velocity?

Additional Problems

1.7 Consider a body of mass m falling near the surface of the Earth and subject to a resistive force that is proportional to the cube of its velocity.

a. Draw a vector diagram of the forces on it.

b. Write Newton's second law for the body and obtain an expression for the terminal velocity. Explain all the units.

c. If the acceleration can be treated as a constant for a short interval of time $\Delta t = 0.05\ \text{s}$, calculate the acceleration, velocity, and position of the body for several intervals up to $t = 0.2\ \text{s}$. Organize the calculations on a table for clarity. Use the numerical values $m = 1$ kg, $v_0 = 0$, $x_0 = 100$ m, $g = 10\ \mathrm{m/s}^2$, and a drag coefficient of $0.5\ \mathrm{Ns^3/m^3}$.

1.8 Two blocks of masses m_1 and m_2 rest on a flat surface, as shown in Figure 1.9, while under an applied force F. The coefficient of static friction between the blocks and the surface is μ_s. Just before motion takes place,

a. Draw a vector diagram showing the forces due to friction as well as the internal body forces, and obtain an expression for the value of F.

b. Write the net force on block 1 and obtain an expression for the internal force exerted on block 1 due to block 2, F_{12}.

c. Write the net force on block 2 and obtain an expression for the internal force on block 2 due to block 1, F_{21}. Finally, evaluate your expressions for F, F_{12}, and F_{21} using $m_1 = 1$ kg, $m_2 = 2$ kg, and $\mu_s = 0.4$.

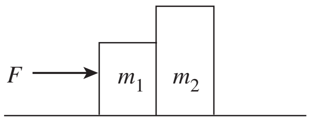

FIGURE 1.9

1.9 Two blocks of masses m_1 and m_2 have an acceleration a, as shown in Figure 1.10, while under an applied force F. The coefficient of kinetic friction between the blocks and the surface is μ_k.

a. Draw a vector diagram showing the forces due to friction as well as the internal body forces, and obtain an expression for the value of F that's responsible for the acceleration of both blocks.

b. Write down Newton's second law for each block.

c. Use the results of Parts (a) and (b) in order to obtain an expression for the value of the internal force between the blocks, say, F_{12}. Finally, evaluate your expressions for F and F_{12} using $m_1 = 2$ kg, $m_2 = 1$ kg, $\mu_s = 0.3$, and $a = 3.5\ \mathrm{m/s}^2$.

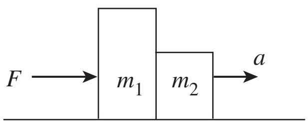

FIGURE 1.10

1.10 A ball is dropped from a height h above the ground. In terms of h obtain expressions for

a. The velocity of the ball just before it hits the ground;

b. The force of impact, if when the ball hits the ground it takes time t_s for it to stop; and

c. The time t_f the ball spends in the air before it reaches the ground.

d. In order for the force of impact to be equal to the ball's weight, what should the value of t_s be, and what can you conclude from this result?

1.11 A spring of stiffness constant k has an unknown mass hanging from its free end. When a mass $m_1 = 2$ kg is added, the spring extends by 30 cm. If a mass $m_2 = 5$ kg is added instead, the spring extends by 85 cm. What are the values of the spring constant and the unknown mass?

1.12 The Verlet algorithm (see Giordano and Nakanishi, 2006) deals with a numerical method of solving a second-order differential equation. In the case of a constant acceleration problem in one dimension, $d^2x/dt^2 = a$, the Verlet method for $x(t)$ results in the discrete formula $x_{i+1} = 2x_i - x_{i-1} + a_i\Delta t^2$, which involves terms to second order in Δt. In Section 1.2, this equation is treated as a system of two first-order differential Equations (1.2.12) with the Euler–Cromer numerical solutions given by (1.2.13 and 1.2.14), wherein (1.2.14) v_{i+1} is used instead of v_i.

a. Show that the Cromer–Euler method is identical to the Verlet algorithm.

b. If in (1.2.14) v_i were to be used rather than v_{i+1}, how would the resulting expression for x_{i+1} compare to that of the preceding Verlet form? (*Hint:* Recall the approximate expression for the derivative Equation [1.2.9].)

2 Application of Newton's Second Law of Motion in One Dimension

2.1 Introduction

In this chapter we consider the application of the general form of Newton's second law of motion, in one dimension, under less general circumstances, those for which the external net force takes on the specific forms:

$$F = Ca = const., \text{ is a constant,} \tag{2.1.1a}$$

$$F = F(t) \Rightarrow a = a(t), \text{ is a function of time,} \tag{2.1.1b}$$

$$F = F(x) \Rightarrow a = a(x), \text{ is a function of position,} \tag{2.1.1c}$$

$$F = F(v) \Rightarrow a = a(v), \text{ is a function of velocity.} \tag{2.1.1d}$$

The importance of these different force behaviors lies in the variable that visibly affects the force. An example of constant acceleration is the case of free fall, under the action of a constant gravitational force ($F = -mg$), which we reviewed in Chapter 1. A common application of a time-dependent force is the case of a driving harmonic force ($F = F_0 \sin(\omega t)$) similar to the one we considered in the forced harmonic oscillator in the previous chapter. Similarly, a force that depends on position can be found in naturally harmonic systems ($F = -kx$) like a pendulum or a spring-mass system. In both cases, the acceleration is proportional to the negative of the displacement. A simple and common example of a velocity-dependent force is an object that falls under the action of gravity with air resistance $(F = -mg - kv)$. As we have seen before, here the force due to air resistance is taken to be proportional to the negative of the velocity raised to some power. Next, we examine each of these situations.

2.2 Constant Force

This represents the simplest application of Newton's second law. We can in fact obtain a general solution for the motion of a particle as follows. We write the acceleration in terms of the net force, taking $F = C$, and

$$a = C/m, \tag{2.2.1a}$$

and using $a = \dfrac{dv}{dt}$, or integrating $\int_{v_0}^{v} dv = \int_0^t a\,dt$, to obtain

$$v = v_0 + at, \tag{2.2.1b}$$

for the velocity as a function of time. Here the object was taken to have an initial velocity v_0 at $t_0 = 0$. Using $v = dx/dt$ and integrating the preceding expression $\int_{x_0}^{x} dx = \int_0^t v dt = \int_0^t \{v_0 + at\}dt$, obtain

$$x = x_0 + v_0 t + \frac{a}{2}t^2, \tag{2.2.1c}$$

with the initial position indicated by x_0. This result is also familiar to us because if we replace the constant acceleration a with the value $-g$ we obtain the general kinematic equations on an object in free fall near Earth's surface. These are the same equations of motion (1.3.4) from Chapter 1, if we make the proper coordinate replacement of $x \to y$. While (2.2.1b) gives the velocity as a function of time, it is instructive to obtain the velocity as a function of position. This is accomplished if we write the acceleration as

$$a = \frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v'\dot{x} = v'v, \tag{2.2.2}$$

where we have now defined the spatial derivative with a prime, $v' \equiv \dfrac{dv}{dx}$, read v-prime, and the time derivative with a dot, $v = \dot{x} \equiv \dfrac{dx}{dt}$, read x-dot. We will use this notation later. Separating variables and integrating, $\int_{v_0}^{v} v dv = \int_{x_0}^{x} a dx$, gives the position-dependent velocity

$$v^2 = v_0^2 + 2a(x - x_0). \tag{2.2.3}$$

We finally notice that if in (2.2.1b) we solve for a then substitute the result into (2.2.1c), we get the complementary useful expression for the position

$$x = x_0 + \frac{v + v_0}{2} t. \tag{2.2.4}$$

EXAMPLE 2.1

In order to take off from a 94.2 m-long runway, a 945.5 kg WW1 SE5a British fighter needed to achieve a speed of 23.7 m/s. If the airplane started from rest and underwent a constant acceleration, what average force and what kind of engine was needed to achieve takeoff?

Solution

From (2.2.3), we can solve for the average force, $C = ma = mv^2/2x$, or $C = 945.5 \text{ kg} \cdot (23.7 \text{ m/s})^2/2 \cdot 94.2 \text{ m} = 2818.9$ N. The time rate of change of energy is the power $\mathrm{p}(t)$. The average power p_{ave} can be obtained by looking at the energy used during takeoff

$$\int_{t_0}^{t} \mathrm{p}(t) dt = \mathrm{p}_{ave} \Delta t = \int_{x_0}^{x} F dx = C \Delta x, \tag{2.2.5a}$$

or

$$\mathrm{p}_{ave} = C \frac{\Delta x}{\Delta t} = C \frac{x - x_0}{t}. \tag{2.2.5b}$$

Solving for $x - x_0$ from (2.2.4) and substituting the result into the preceding expression, we see that the average power under a constant force is also given by

$$\mathrm{p}_{ave} = C\frac{v + v_0}{2}. \tag{2.2.5c}$$

Thus the airplane requires an average power of $\mathrm{p}_{ave} = 2818.9\ \mathrm{N}(23.7\ \mathrm{m/s} + 0\ \mathrm{m/s})/2 = 33404.0\ \mathrm{W}$. A horsepower (hp) is equivalent to 746.3 watts, so this average power corresponds to a 44.8-hp engine. The actual engine used by this kind of aircraft was a Hispano Suiza 200-hp 8-cylinder engine. The difference between our calculated power and the actual engine's power is attributed to the neglect of drag or frictional forces and the assumption that the force is constant throughout. We can get a better value if we assume the engine is about 25% efficient, then using $e = \mathrm{p}_{out}/\mathrm{p}_{in}$, one obtains $\mathrm{p}_{in} = 44.8\text{-hp}/0.25 = 179.2\text{-hp}$, which is closer to the actual value. The actual engine efficiency is probably between 20% and 25%. Finally, notice that (2.2.5c) reduces to $\mathrm{p}_{ave} = Cv$ when the velocity remains constant.

2.3 Time-Dependent Force

In this case, the acceleration can be written as $a(t) = F(t)/m$, and since $a(t)dt = dv(t)$, we can integrate to obtain the velocity

$$\int_{v_0}^{v(t)} dv = \int_0^t a(t)dt \quad \Rightarrow \quad v(t) = v_0 + \int_0^t a(t)dt. \tag{2.3.1}$$

Similarly, since $v(t)dt = dx(t)$, we can integrate this expression to obtain the position as a function of time

$$\int_{x_0}^{x(t)} dx = \int_0^t v(t)dt \quad \Rightarrow \quad x(t) = x_0 + \int_0^t v(t)dt. \tag{2.3.2}$$

To proceed further, knowledge of the actual dependence of $a(t)$ is needed as shown in the following example.

EXAMPLE 2.2

A force $F(t) = F_0 \cos \omega t$ is exerted on a particle of mass m, find analytic expressions for $v(t)$ and $x(t)$. If the particle's initial speed is 0.05 m/s, and it starts from the origin, using values of $m = 1$ kg. $F_0 = 1$ N, and $\omega = 3$ rad/s, give plots of $a(t)$, $v(t)$, and $x(t)$ in the range $0 < t < 10$ s.

Solution

The acceleration is $a(t) = F_0 \cos \omega t / m$, and from (2.3.1) we have for the particle's velocity as a function of time,

$$v(t) = v_0 + \frac{F_0}{m} \int_0^t \cos \omega t \, dt = v_0 + \frac{F_0}{m\omega} \sin \omega t, \tag{2.3.3}$$

which is now used in (2.3.2) to obtain the position as a function of time,

$$x(t) = x_0 + \int_0^t \left\{ v_0 + \frac{F_0}{m\omega} \sin \omega t \right\} dt = x_0 + v_0 t - \frac{F_0}{m\omega^2} (\cos \omega t - 1). \tag{2.3.4}$$

The simple MATLAB script `foft.m` that follows can be used to produce the desired plots shown in Figure 2.1.

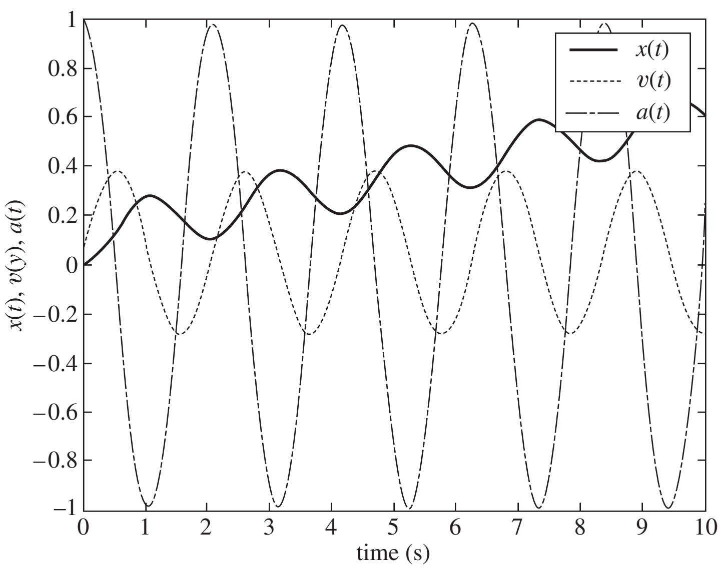

FIGURE 2.1 Example 2.2—Position, Velocity, and Acceleration Plot.

SCRIPT

```
%foft.m
clear;
m=1.0;                              %mass
f0=1.0;                             %force amplitude
w=3.0;                              %force angular frequency
x0=0.0;                             %initial position
v0=0.05;                            %initial velocity
t=[0:0.1:10];                       %time array
a=f0*cos(w.*t)/m;                   %acceleration array
v=v0+f0*sin(w.*t)/m/w;              %velocity array
x=x0+v0.*t-f0*(cos(w.*t)-1)/m/w/w; %displacement array
plot(t,x,'k-',t,v,'b:',t,a,'r-.');
title('x(t),v(t),a(t) due to F=F_0*cos(wt)','FontSize',14)
ylabel('x(t),v(y),a(t)','FontSize',14);
xlabel('time(sec)','FontSize',14);
h=legend('x(t)','v(t)','a(t)'); set(h,'FontSize',14)
```

2.4 Position-Dependent Force

In this case, the net force takes the form $F = F(x) = ma(x)$. We can write the acceleration as

$$a = \dot{v} = \frac{dv}{dx}\frac{dx}{dt} = v'\dot{x} = vv'. \tag{2.4.1}$$

This enables us to write the net force as

$$F(x) = mv(x)\frac{dv(x)}{dx} = \frac{1}{2}m\frac{d}{dx}\left[v(x)^2\right] = \frac{dT}{dx}, \tag{2.4.2}$$

where the mass is assumed constant and where we have used the definition of kinetic energy $T \equiv \frac{1}{2}mv^2$. The preceding expression for the force is significant, because if we separate variables and integrate

$$\int_{x_0}^{x} F(x)\,dx = \int_{T_0}^{T} dT = T - T_0, \tag{2.4.3}$$

which, if we define the work done by a net force on an object displaced from an initial position x_0 to a final position x as

$$W(x) \equiv \int_{x_0}^{x} F(x)\,dx, \tag{2.4.4}$$

then (2.4.3) is a representation of the work–energy theorem,

$$W(x) = \Delta T; \tag{2.4.5}$$

that is, the work done by a net force on an object is equal to the change in kinetic energy of that object. Thus, the approach we will take to analyze the case of a position-dependent force is to view the problem from an energy consideration point of view. It is convenient to define the quantity

$$u(x) \equiv \int_{x_r}^{x} F(x)\,dx, \tag{2.4.6a}$$

as the work done by a force on an object moving from a reference position x_r to position x. This also allows us to write (2.4.4) as

$$W(x) = u(x) - u(x_0). \tag{2.4.6b}$$

Furthermore, we define the potential energy at position x as

$$V(x) \equiv -u(x). \tag{2.4.7}$$

Using this with (2.4.4–2.4.6), we see the change in potential energy,

$$\Delta V(x) = V(x) - V(x_0) = -W(x) = -\int_{x_0}^{x} F(x)\,dx; \tag{2.4.8a}$$

that is, the change in potential energy of an object between initial and final positions equals the negative of the work done by the applied force between the same two points. From (2.4.8) we see that if we define the potential energy such that $V(x_0) = \text{constant} \equiv 0$, then

$$V(x) = -\int_{x_0}^{x} F(x)\,dx, \tag{2.4.8b}$$

which means that $dV(x)/dx = -[F(x) - F(x_0)]$, or

$$F(x) = -\frac{dV(x)}{dx}, \tag{2.4.9}$$

where one has also and consistently taken $F(x_0) = -dV(x_0)/dx = 0$. This indicates that a position-dependent force has a potential energy function associated with it. Looking back to (2.4.3), we see that (2.4.8) implies $\Delta V = -\Delta T$; that is, the sum of potential and kinetic energies

$$V + T = V_0 + T_0 \equiv E, \tag{2.4.10}$$

represents the total *mechanical energy*, E, of a system and that this total energy is constant. This is nothing more than an expression of the conservation of mechanical energy principle.

Given $F(x)$ and using (2.4.9) to obtain the function $V(x)$, the preceding energy concept suggests that one can use (2.4.10) to write $T = \frac{1}{2}m\dot{x}^2 = E - V(x)$, so that

$$\dot{x} = \frac{dx}{dt} = \sqrt{\frac{2[E - V(x)]}{m}}, \tag{2.4.11}$$

where, for convenience, we've taken the positive root. By separating variables and integrating, we find a relationship between position and time as

$$\int_0^t dt = \int_{x_0}^{x} \frac{dx}{\sqrt{\frac{2[E - V(x)]}{m}}} = t. \tag{2.4.12}$$

EXAMPLE 2.3

Consider that a body of mass m moving with velocity v_0 is applied a zero net force. Using Equation (2.4.12), obtain an expression of its position as a function of time.

Solution

Since $F = 0$, Equation (2.4.9) implies that $V(x)$ is a constant, which can be taken to be zero. Equation (2.4.12) simplifies to

$$t = \frac{1}{\sqrt{2E/m}} \int_{x_0}^{x} dx = \frac{x - x_0}{\sqrt{2E/m}}. \tag{2.4.13}$$

Because the initial energy of the body is purely kinetic $E = mv_0^2/2$, which is constant, Equation (2.4.13) gives, $t = (x - x_0)/v_0$, or

$$x = x_0 + v_0 t, \tag{2.4.14}$$

as we expect from Newton's first law for an object moving at constant speed in the absence of an external net force.

EXAMPLE 2.4

Consider a spring-mass simple harmonic oscillator. (a) Assuming the mass is initially at the origin and moving at a speed v_0, use Equation (2.4.12) to obtain an expression for the position of the mass as a function of time. Also, obtain expressions for $v(t)$, $a(t)$, and the total energy of the system. (b) Using the results of Part (a), for one oscillation period, plot the potential energy, the kinetic energy, the total energy, and the spring force as a function of position, and explain their relationship. For plotting purposes, use values of mass, spring constant, and initial velocity of $m = 1$ kg, $k = 0.01$ N/m, $x_0 = 0.0$ m, and $v_0 = 0.5$ m/s respectively.

Solution

(a) For the simple harmonic oscillator spring-mass system, the net force is provided by Hooke's law, $F(x) = -kx$. Using (2.4.9), the potential energy function is

$V(x) = -\int_{x_0}^{x} F(x)dx = \frac{1}{2}k(x^2 - x_0^2)$. Equation (2.4.12) simplifies to

$$t = \int_{x_0}^{x} \frac{dx}{\sqrt{\frac{2}{m}\left[E - \frac{1}{2}kx^2\right]}} = \int_{x_0}^{x} \frac{dx}{\sqrt{\frac{k}{m}\left[\frac{2E}{k} - x^2\right]}}, \tag{2.4.15}$$

since $x_0 = 0$. Making the substitutions $u = x$, $b = \sqrt{2E/k}$, and using the integral of Appendix B, $\int \frac{du}{\sqrt{b^2 - u^2}} = \sin^{-1}(u/b)$, (2.4.15) becomes

$$t = \sqrt{\frac{m}{k}} \int_{x_0}^{x} \frac{dx}{\sqrt{2E/k - x^2}} = \sqrt{\frac{m}{k}} \sin^{-1}\left(\sqrt{\frac{k}{2E}}x\right), \tag{2.4.16}$$

which we can invert to obtain the position as a function of time as

$$x(t) = \sqrt{\frac{2E}{k}} \sin(\omega t) = A \sin(\omega t), \tag{2.4.17}$$

where we have defined $\omega \equiv \sqrt{k/m}$. Also notice from here that $A \equiv \sqrt{2E/k}$ is the maximum vibration amplitude. This result has a harmonic behavior, in agreement with our discussion in Section 1.3 in the absence of damping and a driving force. Furthermore, we can obtain the velocity as a function of time by taking the time derivative to get

$$v(t) = \omega \sqrt{\frac{2E}{k}} \cos(\omega t). \tag{2.4.18}$$

Because initially, at time $t = 0$, the mass is at the origin but moving at v_0, we can solve for the total energy by setting $\omega\sqrt{2E/k} = \sqrt{2E/m} = v_0$, which gives $E = mv_0^2/2$, the initial kinetic energy of the mass. The quantity ω in the argument of the sine and cosine functions is known as the *natural frequency* of the spring-mass system. The period of oscillation is defined to be $\tau = 2\pi/\omega$. Finally, the acceleration is obtained from the force $a(t) = F(x(t))/m = -kx(t)/m$, where $x(t)$ is given by (2.4.17.)

(b) The MATLAB script `fofx.m` with values of $m = 1$ kg, $k = 0.01$ N/m, $x_0 = 0.0$ m, and $v_0 = 0.5$ m/s follows. The actual plots are shown in Figure 2.2, where the total energy, along with the kinetic energy, the potential energy, and the force are shown versus position.

SCRIPT

```
%fofx.m
clear;
m=1.0;                          %mass
k=0.01;                         %spring constant
w=sqrt(k/m);                    %natural frequency
x0=0.0;                         %initial position
v0=0.5;                         %initial velocity
t=[0:0.05:2*pi/w];              %time array from zero to one oscillation period
E0=0.5*m*v0^2;                  %total initial energy
x=sqrt(2*E0/k)*sin(w.*t);       %position versus time array
v=v0*cos(w.*t);                 %velocity versus time array
%a=-k*x/m;                      %acceleration versus time array if needed
PE=0.5*k*x.^2;                  %potential energy array
KE=0.5*m*v.^2;                  %kinetic energy array
E=PE+KE;                        %total energy array
F=-k*x;                         %force array
plot(x,PE,'k-',x,KE,'b:',x,E,'r-.',x,F,'m–');
title('Spring-Mass Simple Harmonic Energy-Force Relation','FontSize',14)
ylabel('PE, KE, E, F','FontSize',14);
xlabel('x(m)','FontSize',14);
h=legend('PE','KE','E','F',3); set(h,'FontSize',14)
```

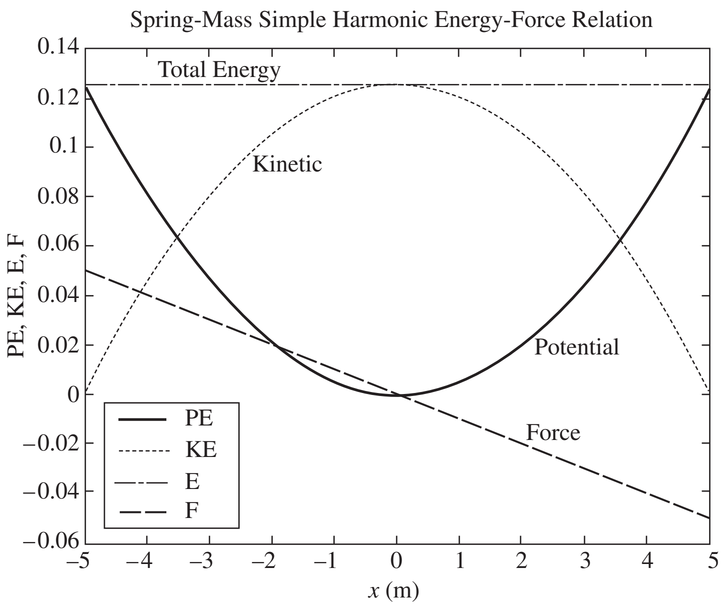

FIGURE 2.2 Example 2.4—Plot of Potential, Kinetic, and Total Energies.

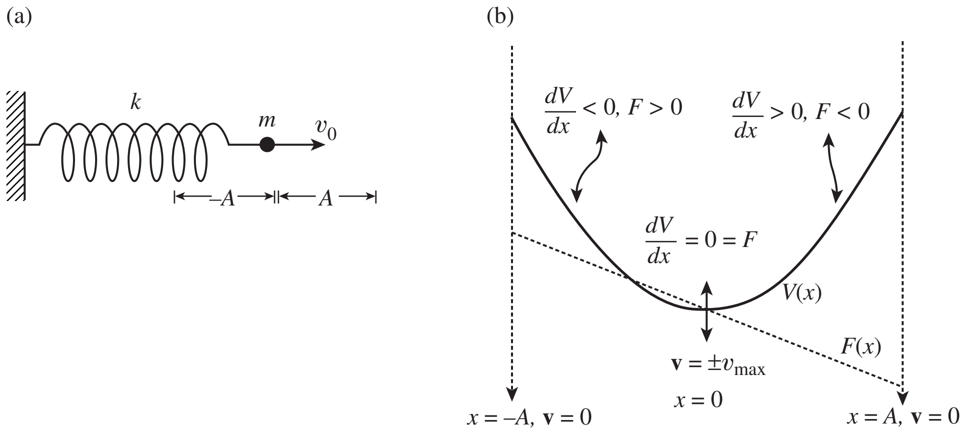

FIGURE 2.3 (a) Mass on a Spring, (b) Potential Energy and Force Diagram.

In Figure 2.3(a) the present initial conditions of the spring-mass system are shown. From the initial conditions given in the problem, we find that $E = mv_0^2/2 = 0.5(1.0)(0.5)^2 = 0.125$ J, which must be conserved. Thus, the amplitude, from (2.4.17) is $A = \sqrt{2E/k} = \sqrt{2(0.125)/0.01} = 5$ m, it follows from (2.4.17), the displacement's range is $-5 \le x \le 5$. As can be seen in Figure 2.2, initially, the energy is all kinetic, but when it's fully stretched at position A, the energy is all potential. After a time equal to half the system's period, the mass returns to its initial position at maximum velocity equal and opposite in direction to its initial value, and the energy is once again all kinetic (KE), but the total energy remains constant. The mass continues until the amplitude is $-A$, at which point the spring is fully compressed and the energy is all potential (PE). The natural frequency of the system has a value of $\omega = \sqrt{k/m} = 0.1$ rad/s. After a full period, $\tau = 2\pi/\omega = 62.832$ s, the mass returns to its initial position ready to begin a new cycle.

Also depicted in Figure 2.2 and in Figure 2.3(b) is the relationship played by the force and the potential energy. This relationship obeys Equation (2.4.9). At the origin, the PE is zero and similarly its derivative, the force is, therefore, zero. The value of x when the force is zero is referred to as the *equilibrium position*. As the spring begins to stretch, the PE begins to increase. The derivative of the potential increases, and by (2.4.9) the force must decrease since $\dfrac{dV(x)}{dx} > 0$, to reach a minimum value when the displacement equals A. At this point the force reaches its most negative value, the spring is fully stretched, which means that the mass experiences a maximum force toward the negative direction. After the maximum displacement is reached, the mass turns around, and begins to return to its equilibrium value, thereby decreasing its PE. The decrease in PE along with a decrease in position corresponds to positive derivative $\frac{dV(x)}{dx} = \frac{-|dV(x)|}{-|dx|} > 0$, which is equivalent to a negative force by (2.4.9), so the mass moves toward $-x$ and begins to gain speed until the force once again reaches zero at the origin, $\frac{dV(x)}{dx} = 0$. The mass continues toward the negative x direction with a maximum velocity of $-v_0$ as it passes through the origin. The PE begins to increase once again as the mass travels toward $-x$. The change in potential is positive, but the change in position is negative; by (2.4.9), since $\frac{dV(x)}{dx} = \frac{|dV(x)|}{-|dx|} < 0$, the force must become positive. This continues to be the trend until the displacement equals $-A$. At this point, the PE is a maximum, the spring is fully compressed, and the force reaches its maximum positive value. This is because the mass experiences the maximum spring force toward the positive direction, begins to turn around, and starts on its return trip toward the origin. In principle, the cycle is repeated indefinitely, unless a damping force is present. All this is shown in Figure 2.3(b). The points of maximum potential energy, i.e., $x = \pm A$, are known as the classical turning points. Finally, notice that Figure 2.2 does confirm that the total energy $E = PE + KE$ is conserved and remains constant throughout the whole cycle.

2.5 Velocity-Dependent Force

As mentioned before, drag forces are velocity-dependent. In such cases we write the force as

$$F = F(v) = m\frac{dv}{dt} \Rightarrow dt = m\frac{dv}{F(v)}, \tag{2.5.1}$$

where the variables have been separated. Integrating the left side on $[0, t]$ and the right side on $[v_0, v]$, we obtain a relationship between velocity and time as

$$t = m\int_{v_0}^{v} \frac{dv}{F(v)}, \tag{2.5.2}$$

which gives the time it takes an object to reach the velocity v, under the action of $F(v)$, if it starts with an initial velocity v_0. If this result is inverted to express $v(t)$, then the position $x(t)$ can be obtained from the now-familiar expression

$$x = x_0 + \int_0^t v(t)\,dt. \tag{2.5.3}$$

An alternate route can be taken if we instead write the force as

$$F(v) = m\frac{dv}{dt} = m\frac{dv}{dx}\frac{dx}{dt} = mv\frac{dv}{dx} \Rightarrow \int_{x_0}^{x} dx = \int_{v_0}^{v} \frac{mvdv}{F(v)} \Rightarrow x = x_0 + m\int_{v_0}^{v} \frac{vdv}{F(v)}, \tag{2.5.4}$$

where the integration has been performed after separating variables. This gives an expression for $x(v)$ that when inverted yields an expression for $v(x)$. To obtain the $x(t)$ we can write $dt = \frac{dx}{v(x)}$, and integrating gives

$$t = \int_{x_0}^{x} \frac{dx}{v(x)}, \tag{2.5.5}$$

which gives an expression for $t(x)$. This resulting expression needs to be inverted to obtain the position as a function of time.

EXAMPLE 2.5

Consider the example of a body of mass m moving near Earth's surface and subject to an air resistive force $R = -Cv$ that was studied numerically in Chapter 1. (a) Obtain analytic expressions for the velocity and position as a function of time. (b) Show that in the limit of zero drag coefficient, the motion has the expected behavior. (c) Give plots of $y(t)$, $v(t)$, $a(t)$ versus t while making sure not to plot beyond the point where $y(t_{\max}) = 0$, where $t_{\max}$ is the time the body reaches the ground. Use the following parameters for plotting purposes: $g = 9.8$ m/s^2, $m = 1$ kg, $C = 0.05$ kg/s, $v_0 = 20$ m/s, and $y_0 = 10.0$ m.

Solution

(a) From Chapter 1, Equation (1.3.5), the velocity-dependent force is written as

$$F(v) = -mg - Cv, \tag{2.5.6}$$

so that from (2.5.2) we have

$$t = m\int_{v_0}^{v} \frac{dv}{-mg - Cv} = -\frac{1}{g}\int_{v_0}^{v} \frac{dv}{1 + \frac{C}{mg}v}. \tag{2.5.7}$$

Making the substitution $u = 1 + \frac{C}{mg}v$, $du = \frac{C}{mg}dv$, we find $t = -\frac{m}{C}\int\frac{du}{u} = -\frac{m}{C}\ln(u)$, or

$$t = -\frac{m}{C}\left[\ln\left(1 + \frac{C}{mg}v\right)\right]_{v_0}^{v} = -\frac{m}{C}\ln\left[\frac{mg + Cv}{mg + Cv_0}\right], \tag{2.5.8}$$

which can be inverted to obtain

$$v = \left(\frac{mg}{C} + v_0\right)e^{-\frac{C}{m}t} - \frac{mg}{C}. \tag{2.5.9}$$

Using (2.5.3) with the coordinate y replacing x, and performing the integration, we obtain the time-dependent position

$$y(t) = y_0 - \left[\frac{m}{C}\left(\frac{mg}{C} + v_0\right)e^{-\frac{C}{m}t} + \frac{mg}{C}t\right]_0^t,$$

or

$$y(t) = y_0 - \frac{mg}{C}t - \frac{m}{C}\left(\frac{mg}{C} + v_0\right)\left(e^{-\frac{C}{m}t} - 1\right). \tag{2.5.10}$$

From (2.5.9) and (2.5.10) we notice that at $t = 0$, the initial conditions $y = y_0$ and $v = v_0$ result, as should be. Also, as $t \to \infty$, the velocity reaches the terminal value $v \to v_t = -mg/C$, as expected from Chapter 1.

(b) In the limit of small drag coefficient, the exponential in (2.5.9) is expanded ($e^{-x} \approx 1 - x + x^2/2 \ldots$) to second order

$$v \approx \left(\frac{mg}{C} + v_0\right)\left(1 - \frac{C}{m}t + \frac{1}{2}\left[\frac{C}{m}t\right]^2\right) - \frac{mg}{C}$$

$$= v_0 - gt - \frac{v_0 C}{m}t + \frac{1}{2}\left(\frac{mg}{C} + v_0\right)\left[\frac{C}{m}t\right]^2, \tag{2.5.11}$$

and in the limit as $C \to 0$ the result is that of the velocity in free fall in the absence of air resistance. Similarly for the position we can write from (2.5.10)

$$y(t) \approx y_0 - \frac{mg}{C}t - \frac{m}{C}\left(\frac{mg}{C} + v_0\right)\left(-\frac{C}{m}t + \frac{1}{2}\left[\frac{C}{m}t\right]^2\right)$$

$$= y_0 + v_0 t - \frac{g}{2}t^2 - \frac{v_0 C}{2m}t^2 \qquad (2.5.12)$$

and as $C \to 0$, there results the familiar expression for the position of an object performing free fall.

(c) For purposes of carrying out the desired plot, the MATLAB script `fofv.m`, shown here, is used. The script calculates the position, velocity, and acceleration of the body using the analytic formulas (2.5.6, 2.5.9, 2.5.10). In order to find the time the body reaches the ground, t_{max}, we solve the equation $0 = y_0 - \frac{mg}{C}t_{max} - \frac{m}{C}\left(\frac{mg}{C} + v_0\right)\left(e - \frac{C}{m}t - 1\right)$ numerically within the script using the MATLAB provided function `fzero`. The solution process requires an initial guess for t_{max}. This is obtained by choosing the positive root in the equation for the position without air resistance, i.e., $0 = y_0 + v_0 t - \frac{g}{2}t_{guess}^2$. This yields $t_{guess} = \frac{v_0}{g} + \sqrt{\left(\frac{v_0}{g}\right)^2 + \frac{2y_0}{g}}$.

SCRIPT

```
%fofv.m
clear;
g=input(' Enter value of gravity g: ');
m=input(' Enter the object mass m: ');
C=input(' Enter the drag coefficient C: ');
if C< 1.e-3 C=1.e-3; end                    %prevent division by zero
y0=input(' Enter the initial height y0: ');  %Initial Conditions
v0=input(' Enter the initial velocity v0: ');%Initial Conditions
%if needed use next line to make NPTS an input
%NPTS=input(' Enter the number calculation steps desired NPTS: ');
NPTS=100;
% use next line to make TMAX a desired input
%TMAX=input(' Enter the run time TMAX: ');
%tz is the time for the frictionless free fall case to have y=0
tz=v0/g+sqrt((v0/g)^2+2*y0/g);
%define a function of several parameters using the inline method
```

```
f=inline('y0-m*(g*t+(m*g/C+v0)*(exp(-C*t/m)-1))/C','t','g','m','C','y0','v0');
%TMAX is the time for the case with drag to reach zero, use tz as guess
TMAX=fzero(f,tz,[],g,m,C,y0,v0);
t=[0:TMAX/NPTS:TMAX];%time array from zero to the time to reach ground
y=y0-m*(g*t+(m*g/C+v0)*(exp(-C*t/m)-1))/C;
v=(m*g/C+v0)*exp(-C*t/m)-m*g/C;
a=-g-C*v/m;
plot(t,y,'k-',t,v,'b:',t,a,'r-.');
title('Resistive Force Proportional to -v Example','FontSize',14)
ylabel('y, v, a','FontSize',14);
xlabel('t(sec)','FontSize',14);
h=legend('y','v','a',0); set(h,'FontSize',14)
```

A plot of the results is shown in Figure 2.4 with inputs of $g = 9.8$ m/s^2, $m = 1$ kg, $C = 0.05$ kg/s, $v_0 = 20$ m/s, and $y_0 = 10.0$ m. Notice that the acceleration is not constant, because the velocity is changing. Just before the body hits the ground, the velocity is about -23 m/s. This value is far from the terminal velocity

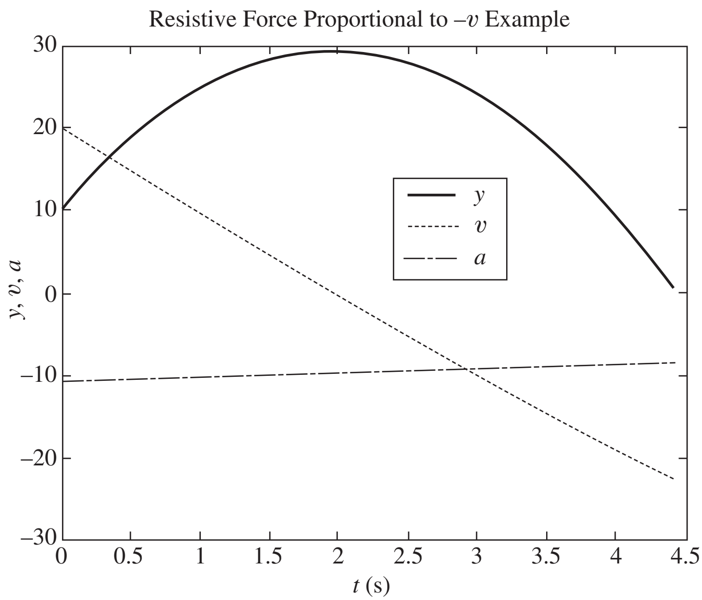

FIGURE 2.4 Example 2.5—Plot of Position, Velocity, and Acceleration for an Object Falling under Linear Air Resistance and Using Analytic Formulas.

value of $v_t = -mg/C = -1*9.8/0.05 = -196$ m/s, at which value the acceleration would approach zero. Here, the body hit the ground before it had a chance to reach the terminal velocity.

Chapter 2 Problems

2.1 What is the average power developed by a 200-kg motorcycle as it applies a constant acceleration from an initial velocity of 0 m/s to 25 m/s in 5 s?

2.2 A truck travels for a distance of 1 km at a constant velocity of 32 m/s. If the truck's engine uses half of its maximum power of 250 hp, what is the average force the engine applies in the process?

2.3 A ball is thrown vertically upward at a speed of 18 m/s.

a. What is the maximum altitude of its flight?

b. What is the ball's time of flight?

2.4 In Example 2.4, what would be a simple way to modify the results obtained for $x(t)$ and $v(t)$ if initially the mass is at rest with the spring stretched by x_0? What is the initial energy?

2.5 Use Equation (2.4.12) to obtain $x(t)$ for an object falling under the action due to gravity near the Earth's surface. Assume it starts from rest.

2.6 **a.** Write the differential equation for $x(t)$ of a spring-mass system performing simple harmonic motion.

b. For this differential equation give a solution, describe its time-dependent behavior, and show that it satisfies the differential equation. Assume that at $t = 0$, $x = x_0$.

2.7 Modify the MATLAB script `fofx.m` in order to obtain a plot of $x(t)$, $v(t)$, and $a(t)$ for the undamped harmonic oscillator and use $x_0 = 0$, $v_0 = 0.5$ m/s for initial conditions. Compare the results with those of the MATLAB script `ho1.m` of Chapter 1 under the same initial conditions.

2.8 A body of mass m initially moving with velocity v_0 experiences a decelerating force $F = -Cv^2$. Find expressions for the body's velocity as a function of time when

a. going down.

b. going up.

c. If the object is dropped from rest, obtain the expression for its position as a function of time. If the object is thrown upward with v_0,

d. what is its position as a function of time,

e. how long does it take the object to reach the maximum height, and

f. what is the maximum height reached?

2.9 Modify the MATLAB script `ho2.m` in order to incorporate the formulas developed in Example 2.5. Compare the numerical and the analytic approaches for the parameters used to obtain Figure 1.7 of Chapter 1. Discuss your results.

Additional Problems

2.10 In the solution for $x(t)$ in Example 2.4 it was assumed that $x_0 = 0$. Redo the problem without this assumption; that is, at $t = 0$, $x = x_0$, and $v = v_0$. Check your results to see that they give the correct limit at $t = 0$.

2.11 When a mass travels under gravity and a drag force due to air resistance, it is possible to model this drag force as $\mathbf{R} = -c\mathbf{v}$; that is, the drag is proportional to the negative of the velocity and c is the drag coefficient. Suppose that under these conditions a mass is thrown directly upward into the air.

a. Obtain an expression for the time (t_{top}) it takes to reach the maximum height.

b. Obtain an approximation of the time to reach the top by expansion methods to first order in the drag coefficient.

c. Show that in the limit of $c \to 0$, the expression for t_{top} simplifies to what is expected in the absence of any drag.

2.12 A ball of mass m falls under gravity while experiencing a force f due to friction. The ball falls from an original height h. It is able to bounce off the floor and reach a height that's $3/4$ of its original height.

a. Obtain an expression for the force f.

b. What is the velocity of the ball when it passes by $1/2$ of the original height after its first rebound?

2.13 A mass m is acted on by the force shown in Figure 2.5. What is the speed of a mass at the end of t_2? (Assume the mass starts from rest.)

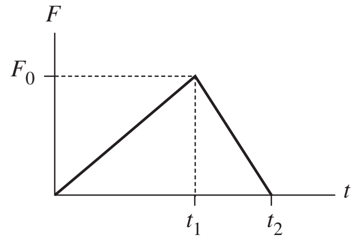

FIGURE 2.5

2.14 A stunt motorcycle rider traveling with speed v_0 suddenly deploys a parachute. The decelerating force experienced is proportional to the square of his velocity.

a. Give an expression for the velocity of the motorcycle as a function of time (let $t = 0$ be the time at which the parachute is deployed).

b. Show that at $t = 0$, and your expression gives the correct limit.

c. How long would it take before his velocity is one half of his initial velocity?

2.15 Find the maximum height an object thrown upward with an initial speed v_0 will reach if it is subject to gravity and to air resistance proportional to its velocity.

2.16 A spherical marble of radius a in a fluid experiences a drag force given by Stokes' law, $\mathbf{R} = -6\pi\eta a\mathbf{v}$, where η is the fluid viscosity and $\mathbf{v}$ its velocity. Suppose that this marble is placed on the surface of the ocean and let go.

a. Write down Newton's second law for the marble.

b. Derive an expression for the terminal speed of the marble in terms of its density ρ_0, its radius, the ocean water density ρ_w, gravity g, and the viscosity.

2.17 A force $F(t) = F_0 \cos \omega t + ma_0$ is exerted on a particle of mass m.

a. Find analytic expressions for $v(t)$ and $x(t)$.

b. If the particle's initial speed is 0.05 m/s, and it starts from the origin, using values of $m = 1$ kg, $F_0 = 1$ N, $a_0 = 0.02$ m/s^2, and $\omega = 3$ rad/s give plots of $a(t)$, $v(t)$, and $x(t)$ in the range $0 < t < 10$ s. Explain the observed behavior of the motion.