Project for Dennis Zill and Michael Cullen's Advanced Engineering Mathematics
Vibration Control: Vibration Isolation (for Chapter 3)

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An important part of mechanical engineering is the suppression of unwanted vibrations in machinery and structures. While this is an active area of research and development, the basic principles behind the most commonly used vibration control strategies date back over 100 years and can be illustrated with very simple spring/mass models. Moreover, the analysis of these models requires nothing beyond the basic theory of second order linear constant coefficient differential equations. Here we investigate a strategy for vibration control that can be modelled using a single equation. In a later project, you can investigate a different strategy that is modelled by a system of equations.

Suppose that a machine (for example an automobile engine) that vibrates in the course of its operation must be mounted on some sort of base. If the machine is mounted rigidly onto the base, its vibrations will be transmitted to and through the base. This might cause a number of problems, including damage to the base or the mounts or unacceptable discomfort for people using the machine.

A natural way to try to minimize such unwanted effects is to insert a protective mounting apparatus between the machine and the base. (In the case of an automobile engine, the engine is attached to the car not rigidly, but with parts called motor mounts.) Such a protective device, called a **vibration isolator**, commonly consists of one or more pads of elastic material (rubber, cork, or more exotic elastomers), a set of coil springs (such as is found inside a mattress), a sealed chamber containing air under pressure (commonly referred to as an air spring), or some combination of these. Regardless of the details of its design (and there have been hundreds of patents granted for such designs), a vibration isolator can be idealized by the configuration indicated in Figure 1(a). The machine is represented by a mass m, the elasticity of the isolator by a spring having spring constant k, and the damping of the isolator by a dashpot having damping constant β. This is merely the spring/mass system analyzed in Section 3.8. The equation governing the motion of the mass m is:

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \beta\frac{\mathrm{d}x}{\mathrm{d}t} + kx = F_0 \sin\omega t \tag{1}$$

where x represents the displacement of the mass from its equilibrium position, t is time, and the vibration of the machine is represented by a sinusoidal forcing term of the form $F_0 \sin\omega t$.

We will quantify the effectiveness of this arrangement by its **transmissibility** T:

$$T = \frac{\text{maximum force transmitted to the base through the vibration isolator}}{\text{maximum force exerted on the base by a rigidly attached machine}} = \frac{F_T}{F_0}.$$

Since the transient terms in the general solution of the above equation decay exponentially with time, we will calculate the transmissibility based on a steady state solution of (1). Ultimately, we wish to understand how the transmissibility depends on the parameters k and β.

1. Recall that a steady state solution can be found in the form

$$x_s = c_1 \cos\omega t + c_2 \sin\omega t = A\sin(\omega t + \phi) \tag{2}$$

where $A = \sqrt{c_1^2 + c_2^2}$. By substituting the first of these forms into (1), show that c_1 and c_2 satisfy the system of equations

$$\begin{aligned} (k - m\omega^2)c_2 - \beta\omega c_1 &= F_0 \\ \beta\omega c_2 + (k - m\omega^2)c_1 &= 0 \end{aligned}$$

and that

$$A = \frac{F_0}{\sqrt{(\beta\omega)^2 + (k - m\omega^2)^2}}.$$

2. The force exerted on the base through the vibration isolator is $\beta \frac{\mathrm{d}x_s}{\mathrm{d}t} + kx_s$. Use the second form of x_s in (2) to show that the maximum magnitude of this force is

$$F_T = A\sqrt{k^2 + (\beta\omega)^2}$$

and that the transmissibility can be expressed as

$$T = \sqrt{\frac{1 + \left(2\frac{\beta}{\beta_c}\frac{\omega}{\omega_n}\right)^2}{\left(1 - \left(\frac{\omega}{\omega_n}\right)^2\right)^2 + \left(2\frac{\beta}{\beta_c}\frac{\omega}{\omega_n}\right)^2}}$$

where $\omega_n = \sqrt{k/m}$ is the natural frequency of the spring, and $\beta_c = 2\sqrt{mk} = 2m\omega_n$ is the value of β giving critical damping in (2).

Note that we have expressed T in terms of two dimensionless quantities: β/β_c and ω/ω_n. So we can use this expression to quantify the dependence of T on ω (that is, the effectiveness of the vibration isolator as a function of the frequency of the machine vibration) in a way that makes no reference to an arbitrary choice of units.

3. Use a graphing utility to graph T as a function of ω/ω_n for $\beta/\beta_c = 0$, 1/4, 1/2, 3/4, and 1, all on the same set of axes. (A viewing window of $0 \le \omega/\omega_n \le 3$ by $0 \le T \le 3$ will show the relevant features of these four graphs.)

4. Much can be deduced from these graphs, some of which may not be intuitively obvious and may even seem counterintuitive.

(a) Verify both graphically and algebraically that the point $(\sqrt{2}, 1)$ appears on each graph regardless of the value of β/β_c.

(b) For what values of ω/ω_n does the vibration isolator reduce the maximum value of the force transmitted to the base ($T < 1$)? Are there values of ω/ω_n for which the vibration isolator makes things worse ($T > 1$)? Your answers should be independent of β/β_c.

(c) Verify your answers to (b) algebraically by working with the inequalities $T < 1$ and $T > 1$.

(d) The incorporation of some damping ($\beta > 0$) protects the base against the resonance that would occur for $\beta = 0$ if ω should, temporarily or by some accident, drift near ω_n. For fixed values of ω and ω_n such that ω/ω_n is within the range determined in (b), does increasing the damping constant β of the vibration isolator (and thereby also increasing β/β_c) improve or detract from its effectiveness?

(e) Summarize your findings for the working engineer. Should the spring constant k of the vibration isolator be large (corresponding to "hard" springs) or small (corresponding to "soft" springs)? What would you advise about the degree of damping (i.e., the size of β)?

The same sort of vibration isolator can also be employed in the opposite sense to that considered above: to protect an object mounted on a base from vibrations of the base. For example, a CD player mounted in an automobile needs to be protected from vibrations of the automobile. On a larger scale, entire buildings have been built on foundations incorporating vibration isolators. To model this situation, refer to Figure 1 (b) where x represents the vertical displacement of the object (mass m) from its equilibrium position and $y = Y_0 \sin\omega t$ represents a sinusoidal motion of the base so that

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -k(x - y) - \beta\frac{\mathrm{d}}{\mathrm{d}t}(x - y). \tag{3}$$

5. Substitute the postulated form for y into (3) to produce a differential equation for x of the form

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \beta\frac{\mathrm{d}x}{\mathrm{d}t} + kx = Y_0\sqrt{k^2 + (\beta\omega)^2}\sin(\omega t + \psi).$$

Then show that a steady state solution of this last equation can be found in the form

$$x_s = TY_0\sin(\omega t + \psi + \phi)$$

where T is as above. (Hint: most of the work was already done in Problem 1!)

Thus the transmissibility T is also a measure of the effectiveness of a vibration isolator from this alternate point of view, but in reference to the amplitude of the motion transmitted from a base to a mounted object instead of the amplitude of the force transmitted from a mounted machine to a base.

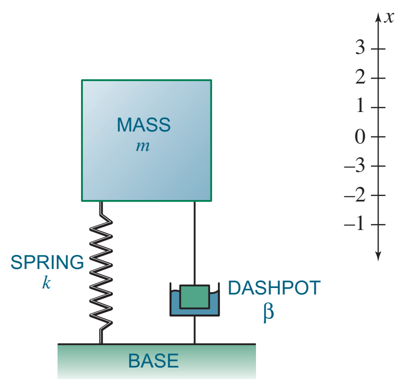

Figure 1 (a)

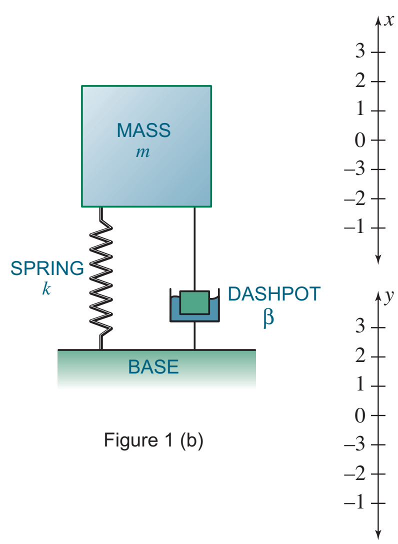

Figure 1 (b)