Project for Section 16.2
Numerical Simulation of Optical Bistability

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1 Problem setup

At the end of 20th century, George Sinitsyn and his coworkers [1] discovered that all-optical shift register could direct the binary data flow perpendicularly to the light beam. This remarkable property was due to the phenomenon, called optical bistability or optical hysteresis. Optical hysteresis and transverse effects of optical bistability can surprisingly be a basis in developing novel systems for optical computing and processing.

One of the prospective devices for realization of optical bistable response is a Fabry-Perot interferometer with strong nonlinearity of the spacer between the mirrors.

Light propagation in optically nonlinear material of the spacer causes a change in its refractive index. In turn, the transmittance of the interferometer depends on the refractive index in such a way that when the light intensity reaches a certain value, the transmittance jumps rapidly up and light goes through almost without losses. This process is referred as optically controlled switching-on. Decreasing of the input light intensity leads to drop in the interferometer transmittance back to initial state. But switching-on and switching-off take place at different input intensities. Such phenomenon of observing the high transparency at one light beam power while increasing and breaking transparency at another power while decreasing can be considered as optical hysteresis.

The model is based on the non-linear heat transfer equation

$$\rho c \frac{\partial u}{\partial t} = \frac{k}{r}\frac{\partial}{\partial r}\left(r\,\frac{\partial u}{\partial r}\right) + Q\,(t,r,u) - \rho c\,\frac{u}{\tau}, \tag{1}$$

where u is the parameter of the medium that contributes to nonlinear change of refractive index (for instance, it can be carrier density, temperature, pressure, etc.); t is time, r is the distance in polar coordinates, ρ, c, and k are some physical parameters of the medium, in particular, k determines the diffusion of the corresponding parameter u, Q is a source function for u, and τ is time constant describing the relaxation of u.

When this model is applied to thin-film semiconductor interferometers that are characterized by thermal optical nonlinearity, then u is the excess temperature of the intermediate nonlinear layer, k is the thermal conductivity, ρ is the density, c is the specific heat of the material, and Q represents the heat released in the intermediate layer as a result of light absorption.

Suppose that the nonlinear Fabry-Perot interferometer has the radius r_M and thickness of spacer h ($h \ll r_M$). Assume also that incident light intensity distribution is modeled by the formula:

$$I_i(t,r) = I_0(t)\Big/ e^{r^2/r_f^2} \qquad (h \ll r_f \ll r_M), \tag{2}$$

where

$$I_0(t) = I_i(t,0) = \begin{cases} I_m\, d\,t, & \text{if } \ 0 \le t \le t_0, \\ I_m\, d\,(2t_0 - t), & \text{if } \ t_0 < t \le 2t_0. \end{cases} \tag{3}$$

In this case, the source function is

$$Q(t,r,u) = \frac{(1-R^2)\,\alpha}{(1-R)^2 + 4R\,\sin^2\left\{\pi\left[\varepsilon + \frac{m+\varepsilon}{n_0}\,\delta n(u)\right]\right\}} \cdot I_i(t,r); \tag{4}$$

the interferometer transmittance is

$$T(t,r) = \frac{(1-R)^2 \cdot e^{-\alpha h}}{(1-R\cdot e^{-\alpha h})^2 + 4R\cdot e^{-\alpha h}\,\sin^2\left\{\pi\left[\varepsilon + \frac{m+\varepsilon}{n_0}\,\delta n(u)\right]\right\}}; \tag{5}$$

the transmitted light intensity distribution becomes

$$I_T(t,r) = T(t,r)\cdot I_i(t,r). \tag{6}$$

Here r_f is the radius of the input light beam at intensity level $I_i(t,r) = I_0(t)/e$, t_0 is the moment at which the intensity $I_0(t)$ runs up to maximum I_m, d is speed factor of input light intensity changing, R is mirror reflectivity, α is absorption factor, m is the order of interference, n_0 is the initial refractive index, $\delta n(u) = n_u u$ is the change of refractive index associated with the change of the temperature u, n_u is thermo-optical coefficient; $\varepsilon = H(1-R)/\pi\sqrt{R}$ is the value of detuning from the interference resonance, H is detuning parameter.

Assuming that the interferometer is insulated at $r = r_M$, we get the following boundary and initial conditions:

$$u|_{r=0} < \infty, \qquad \left.\frac{\partial u}{\partial r}\right|_{r=r_M} = 0, \qquad u|_{t=0} = 0. \tag{7}$$

Our goal is to calculate numerically the dependence $I_T(t,0)$ as a function of $I_i(t,0)$.

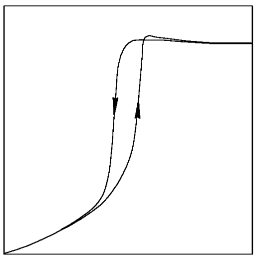

Figure 1: Hysteresis.

One of the value set for physical parameters may be chosen a follows:
$c = 0.71$ J/(g · K), $\rho = 2.202$ g/sm^3, $k = 0.0119$ W/(sm · K) $\tau = 10^{-4}$ s, $r_M = 0.2$ sm, $R = 0.976$, $m = 2$, $n_0 = 2.3$, $\alpha = 20$ sm^{-1}, $h = 2.75\cdot 10^{-5}$ sm, $H = 1$, $n_u = 10^{-4}$, $I_m = 10^4$ W/sm^2 $r_f = 2\cdot 10^{-3}$ sm, $d = 100$ s^{-1}, $t_0 = 0.02$ s.

2 Discretization of linear part

To solve the initial boundary value problem (1) – (7) numerically, we start with its linear part in Eq. (1). Its approximation is based on the implicit finite difference scheme of Crank-Nicholson, as it is described in the text, §16.2. First, we define the mesh points in the rt-plane by intersection of the circles $r_m = m\Delta r$,

$m = 1, 2, \dots M$ and the lines $t = n\Delta t$, $n = 0, 1, 2, \dots N$ and denote the approximate values of u at these points by u_m^n, Δr and Δt being the mesh spacings in the space and time directions, respectively.

The derivatives with respect to r we approximate using the second order accurate finite difference formula:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\,\frac{\partial u}{\partial r}\right) = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} \sim \frac{u_{m+1} - 2u_m + u_{m-1}}{(\Delta r)^2} + \frac{1}{r_m}\,\frac{u_{m+1} - u_{m-1}}{2(\Delta r)}.$$

Then the Crank-Nicholson approximation to the linear part in the equation (1) becomes

$$\begin{aligned}\frac{u_m^{k+1} - u_m^k}{\tau} =& \frac{a^2}{2}\left[\frac{u_{m+1}^{k+1} - 2u_m^{k+1} + u_{m-1}^{k+1}}{(\Delta r)^2} + \frac{1}{r_m}\,\frac{u_{m+1}^{k+1} - u_{m-1}^{k+1}}{2(\Delta r)}\right] \\ &+ \frac{a^2}{2}\left[\frac{u_{m+1}^{k} - 2u_m^{k} + u_{m-1}^{k}}{(\Delta r)^2} + \frac{1}{r_m}\,\frac{u_{m+1}^{k} - u_{m-1}^{k}}{2(\Delta r)}\right],\end{aligned} \tag{8}$$

where $a^2 = k/\rho c$ is the thermal diffusivity. Various schemes are available for finite-difference approximation of nonlinear heat-transfer problems as a system of linear algebraic equations. They include, among others the linearization procedures (that are not applicable in our case due the nature of the nonlinear term) and the lagging [2].

We can rewrite the scheme (8) as

$$u_{m+1}^{k+1}\left(\frac{a^2(\Delta t)}{2(\Delta r)^2} + \frac{a^2(\Delta t)}{4r_m\,(\Delta r)}\right) + u_m^{k+1}\left(1 - \frac{a^2(\Delta t)}{(\Delta r)^2}\right) + u_{m-1}^{k+1}\left(\frac{a^2(\Delta t)}{2(\Delta r)^2} - \frac{a^2(\Delta t)}{4r_m\,(\Delta r)}\right) + \dots,$$

where by dots $(\dots)$ we denote all known terms at $t = k(\Delta t)$. We can rewrite this system of algebraic equations in a matrix form for the $(M+1)$-unknown vector-column $\mathbf{u}^T = (u_0^{k+1}, u_1^{k+1}, \dots u_M^{k+1})$:

$$\mathbf{A}\,\mathbf{u}^{k+1} = \mathbf{f}(\mathbf{u}^k), \tag{9}$$

where $\mathbf{A}$ is the tridiagonal matrix:

$$\mathbf{A} = \begin{bmatrix} a_0 & c_0 & 0 & 0 & \dots & 0 \\ b_1 & a_1 & c_1 & 0 & \dots & 0 \\ 0 & b_2 & a_2 & c_2 & \dots & 0 \\ \vdots & & \ddots & & & \vdots \\ 0 & \dots & & b_{M-1} & a_{M-1} & c_{M-1} \\ 0 & \dots & & 0 & b_M & a_M \end{bmatrix}. \tag{10}$$

The elements of the vector-column $\mathbf{f}$ are obtained from the triangulated expression:

$$f(u) = \frac{a^2}{2}\left[\frac{u_{m+1}^{k} - 2u_m^{k} + u_{m-1}^{k}}{(\Delta r)^2} + \frac{1}{r_m}\,\frac{u_{m+1}^{k} - u_{m-1}^{k}}{2(\Delta r)}\right] + Q(t, r, u)/\rho c - u/\tau.$$

The coefficients of the matrix $\mathbf{A}$ are defined as follows:

$$c_m = \frac{a^2(\Delta t)}{2(\Delta r)^2} + \frac{a^2(\Delta t)}{4r_m\,(\Delta r)}, \quad a_m = 1 - \frac{a^2(\Delta t)}{(\Delta r)^2}, \quad b_m = c_m - \frac{a^2(\Delta t)}{2r_m\,(\Delta r)}, \quad 1 \le m \le M-1.$$

We are left to determine the values of coefficients a_0, c_0, b_M, and a_M from the boundary conditions. At $r = 0$ we have a singularity, so determine its value we use l'Hopital's rule:

$$\lim_{r\to 0}\frac{1}{r}\frac{\partial}{\partial r}\left(r\,\frac{\partial u}{\partial r}\right) = \lim_{r\to 0}\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right) = \lim_{r\to 0}\left(\frac{\partial^2 u}{\partial r^2}\right) + \lim_{r\to 0}\frac{\partial u/\partial r}{r} = \lim_{r\to 0} 2\,\frac{\partial^2 u}{\partial r^2}.$$

To use a second order accurate central difference formula at $r = 0$, a node is needed to the left of the origin $r = 0$. This is achieved by considering a fictitious node # "-1" at a fictitious temperature u_{-1} located at a distance Δr to the left of the r axis. The resulting finite difference approximation at $r = 0$ becomes

$$2\,\frac{\partial^2 u}{\partial r^2} \sim 2\,\frac{u_{-1} - 2u_0 + u_1}{(\Delta r)^2}.,$$

where the fictitious u_{-1} is determined by utilizing the symmetry condition at the node $m = 0$; that is,

$$\left.\frac{du}{dr}\right|_{r=0} = \frac{u_1 - u_{-1}}{2h} = 0 \qquad \Longrightarrow \qquad u_1 = u_{-1}.$$

Similarly we can approximate the flux boundary condition $\partial u/\partial x|_{r=r_M}$, which yields the system of $M + 1$ algebraic equation with $M + 1$ unknows.

3 Numerical method

In order to solve tri-diagonal system of equations (9) – (10) resulting from the discretization, we consider a LU decomposition method which requires in this case $O(n)$ numerical operations only.

Define $\mathbf{A} = LU$ and rewrite the system as $LUu = f$, where

$$L = \begin{bmatrix} \alpha_0 & 0 & & \dots & 0 \\ b_1 & \alpha_1 & 0 & \dots & 0 \\ 0 & b_2 & \alpha_2 & \dots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & \dots & & b_M & \alpha_M \end{bmatrix}, \qquad U = \begin{bmatrix} 1 & \gamma_0 & & \dots & 0 \\ 0 & 1 & \gamma_1 & \dots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & \dots & & 1 & \gamma_{M-1} \\ 0 & \dots & & 0 & 1 \end{bmatrix}.$$

After multiplying we can write:

$$\begin{aligned} a_0 &= \alpha_0, \quad \alpha_0\gamma_0 = c_0 \\ a_i &= \alpha_i + b_i\gamma_{i-1}, \quad i = 1, 2, \dots, M \\ \alpha_i\gamma_i &= c_i, \quad i = 1, 2, \dots, M-1 \end{aligned} \tag{11}$$

Using (11) we derive the recursive formulae for coefficients of L and U:

$$\begin{aligned} \alpha_0 &= a_0, \quad \gamma_0 = \frac{c_0}{\alpha_0} \\ \alpha_i &= a_i - b_i\gamma_{i-1}, \quad i = 1, 2, \dots, M-1 \\ \alpha_M &= a_M - b_M\gamma_{M-1} \end{aligned} \tag{12}$$

To solve $LUu = f$, let $Uu = z$ and $Lz = f$. Then

$$\begin{aligned} z_0 &= \frac{f_0}{\alpha_0}, \quad z_i = \frac{f_i - b_i z_{i-1}}{\alpha_i}, \quad i = 1, 2, \dots, M \\ u_M &= z_M, \quad u_i = z_i - \gamma_i u_{i+1}, \quad i = M-1, M-2, \dots, 0 \end{aligned} \tag{13}$$

References

[1] S.P. Apanasevich, A.V. Lyakhovich, G.V. Sinitsyn. 2D all-optical shift registers: numerical simulation. In *Optical Computing, Inst. Phys. Conf.* 1995, No. 139, part 1, pp. 109 – 112.

[2] M.Necati Ozisik. *Finite Difference Methods in Heat Transfer.* CRC Press, 1994.