▸▸▸ PROJECT FOR SECTION 9.16

# Minimal Surfaces

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If you dip a wire frame into a soap solution and carefully remove it, you will find a soap film stretched across the wire. If the wire frame is planar, like the circular rings typically used to blow bubbles, then the soap film will be flat. Frames bent into more interesting shapes, however, yield more interesting surfaces.

A legendary figure in the study of such shapes was the Belgian physicist Joseph Plateau (1801–1883). Although blind (as a result of staring into the Sun for 25 seconds as an experiment in the physiology of vision), he directed an extensive series of experiments with soap films using a special solution of glycerin and soap of his own devising with which he could make soap films that could last for hours. Plateau also worked extensively with soap bubbles. (Through painstakingly careful observations, he was able to conjecture some beautifully simple principles governing the geometry of clusters of soap bubbles known as "Plateau's rules.")

Plateau realized that a soap film forms so as to minimize its energy due to surface tension or, equivalently, to minimize its surface area subject to the constraint that it spans the wire. He challenged mathematicians to give a general description of such area-minimizing surfaces, or **minimal surfaces**. As a consequence, the problem of determining the surface of least area constrained by a given boundary is known as "Plateau's problem."

At the time of Plateau, the mathematical study of minimal surfaces had already begun almost a century earlier with work by Leonhard Euler and Joseph Louis Lagrange. The mathematics necessary to resolve many of the conjectures and problems of Plateau did not develop until the twentieth century. Indeed, the study of minimal surfaces remains an active area of research today, and mathematicians still scramble to keep up with its applications and potential applications.

Applications abound in many of the physical and biological sciences. Much recent excitement centers on applications to nanotechnology in molecular engineering and materials science. Some very exotic minimal surfaces, only recently discovered mathematically, have been observed physically in "block copolymers," molecules composed of two different polymer strands that repel each other. The molecules arrange themselves in such a way that the boundaries between the dissimilar parts form minimal surfaces. This case is a typical application in the sense that the interface between any two substances that repel each other tends to be, at least approximately, a minimal surface.

More esoteric applications include the general relativistic description of black holes. There are also applications in design. For example, engineers sometimes use minimal surfaces to design structures over which stress should be distributed as uniformly as possible to maximize durability. Finally, minimal surfaces are aesthetically pleasing and are often used in architecture and art, including the sculptures of the well-known mathematician-artist Helaman Ferguson.*

Let's consider a simple version of **Plateau's problem**:

> Let $R$ be a closed and bounded region in the $xy$-plane bounded by a piecewise smooth simple closed curve $C$. Let $z = g(x, y)$ be a given function defined on $C$. (The graph of $g$ is our "wire frame.") Among all functions $z = u(x, y)$ having continuous second partial derivatives on $R$ and such that $u(x, y) = g(x, y)$ on $C$, characterize the one whose graph over $R$ has the smallest possible surface area.

In our attempt to solve the foregoing problem, we begin with (2) in Definition 9.11 of the text. The surface area $A$ of the graph of $u$ over $R$ is given by

$$A(u) = \iint_R \sqrt{1 + [u_x(x, y)]^2 + [u_y(x, y)]^2}\, dA$$
$$= \iint_R \sqrt{1 + \|\nabla u(x, y)\|^2}\, dA.$$

Now take any function $w(x, y)$ such that $w = 0$ on $C$ and consider the following real-valued function: $F(t) = A(u + tw)$ for small values of $t$. If $u$ is the function that minimizes $A$ over all functions having the values prescribed by $g$ on $C$, then $t = 0$ is a critical value for $F$; that is, $F'(0) = 0$. Note that

$$F'(t) = \frac{d}{dt}\iint_R \sqrt{1 + \|\nabla u + t\nabla w\|^2}\, dA$$
$$= \iint_R \frac{\partial}{\partial t}\sqrt{1 + \|\nabla u + t\nabla w\|^2}\, dA$$

*For other surfaces, see www.helasculpt.com/gallery.

## Related Problems

1. Use the definition of norm in terms of the dot product to show that

$$F'(0) = \iint_R \frac{\nabla u}{\sqrt{1 + \|\nabla u\|^2}} \cdot \nabla w \, dA.$$

2. Suppose that $h$ is a function and $\mathbf{F}$ is a vector field defined on $R$ such that the first partial derivatives of $h$ and the two component functions of $\mathbf{F}$ are continuous on $R$. Use the vector identity

$$\text{div } (h\mathbf{F}) = h \text{ div } \mathbf{F} + (\text{grad } h) \cdot \mathbf{F}$$

(Problem 27, Exercises 9.7) and the alternative form of Green's theorem given in (1) of Section 9.16 to show that

$$\oint_C (h\mathbf{F} \cdot \mathbf{n}) \, ds = \iint_R (h \text{ div } \mathbf{F} + (\text{grad } h) \cdot \mathbf{F}) \, dA.$$

3. Apply this last identity to the result given in Problem 1 to show that

$$\iint_R w \text{ div}\left(\frac{\nabla u}{\sqrt{1 + \|\nabla u\|^2}}\right) dA = 0.$$

Because this is true for any function $w(x, y)$ such that $w = 0$ on $C$, it must be the case that

$$\text{div}\left(\frac{\nabla u}{\sqrt{1 + \|\nabla u\|^2}}\right) = 0.$$

4. Show that the last equation in Problem 3 can be expressed as the nonlinear partial differential equation

$$(1 + u_y^2)u_{xx} + (1 + u_x^2)u_{yy} - 2u_x u_y u_{xy} = 0.$$

This equation, which is known as the **minimal surface equation**, was first written down by Lagrange in 1760.

5. Show that if $u$ is a function of $x$ only or $y$ only, then the graph of $u$ is a plane.

6. Use the Chain Rule and polar coordinates to show that if $u = f(r)$, then

$$rf''(r) + f'(r)(1 + [f'(r)]^2) = 0$$

7. The second-order ODE in Problem 6 is a separable first-order ODE in $f'(r)$. Use the method of Section 2.2 to show that if $u = f(r)$, then

$$\frac{du}{dr} = \frac{1}{\sqrt{r^2/c^2 - 1}}.$$

Use the substitution $r = c \cosh u$ to show that $r = c \cosh\left(\frac{u - d}{c}\right)$, where $c$ and $d$ are constants.

Note that this is the surface obtained by revolving a catenary (see Section 3.10) around the $z$-axis. The surface of revolution is known as a **catenoid**. The catenoid was the first nonplanar minimal surface ever described (by Euler in about 1740). A soap film formed between two coaxial rings takes on this shape, not the shape of a cone or cylinder! See Figure 1.

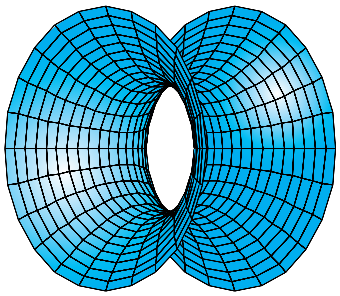

**Figure 1** Catenoid

8. Use the Chain Rule and polar coordinates to show that if $u = f(\theta)$, then $u = c\theta + d$, where $c$ and $d$ are constants. This surface—the spiral traced out by a horizontal line rotating around the $z$-axis with constant angular velocity while rising along the $z$-axis with constant velocity—is known as the **helicoid**. It was the second nonplanar minimal surface ever described (by Jean Babtiste Meusnier in 1776). From Figure 2 you might recognize the helicoid as a model for the rotating curved blades in machinery such as post hole diggers, ice augers, and snow blowers.

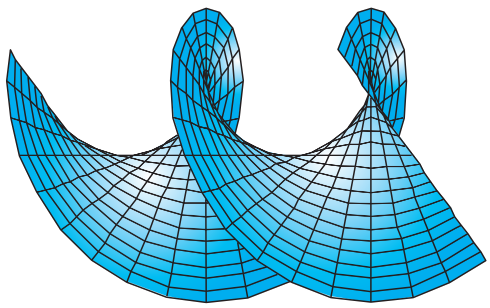

**Figure 2** Helicoid

## Afterword

Most minimal surfaces are geometrically more complicated than the catenoid and the helicoid and can be represented conveniently only in parameterized form rather than as graphs of functions. The study of the parameterizations of minimal surfaces has deep connections with harmonic functions and complex analysis, the subject of Part 5 of this text.