An Introduction to Euclidean Three Dimensional Geometry
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by Shlomo Libeskind.
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# An Introduction to Euclidean Three Dimensional Geometry

In the Appendix on Basic Notion we listed the following Axioms:

Axiom 1: Lines, planes and space are sets of points. Space contains all points.

Axiom 2: Two distinct points are on exactly one line.

Axiom 3: Any three noncollinear points determine a unique plane.

Axiom 4: If two points on a line are in a plane, then the entire line is in the plane.

Axiom 5: If two distinct plans have a point in common, then the planes have an entire line in common.

**NOW TRY THIS 1-1**: Prove that two distinct planes that have a point in common have exactly one line in common.

The last three axioms are the foundation for three dimensional geometry. We have seen that in the plane two lines could either intersect in a single point or be parallel. In space we have the following basic relation between two lines:

(a) Two lines may **intersect** in a single point.

(b) Two lines are **parallel** if there is a plane that contains them and they have no point in common. (This is the definition of parallel lines).

(c) Two lines are **skew lines** if there is no single plane that contains them (i.e. they are not coplanar).

**Example 1-1.** In Figure 1-1 in which point $D$ is not in plane $\alpha$ and $A, B, C$ are not collinear points in $\alpha$, we see an example of skew lines: lines $\overleftrightarrow{BC}$ and $\overleftrightarrow{AD}$ are skew. To prove that the lines are skew we need to show that there is no single plane that contains them. Suppose otherwise, i.e., that $\overleftrightarrow{BC}$ and $\overleftrightarrow{AD}$ are both in same plane $\beta$ (not shown). Then points $A, B$ and $C$ are in that plane. Since planes $\alpha$ and $\beta$ both contain $A, B$ and $C$, by Axiom 3, $\beta = \alpha$. This implies that $D$ is in plane $\alpha$ contrary to our choice of $D$ outside of $\alpha$. Consequently $\overleftrightarrow{AD}$ and $\overleftrightarrow{BC}$ are not coplanar and therefore skew. ☐

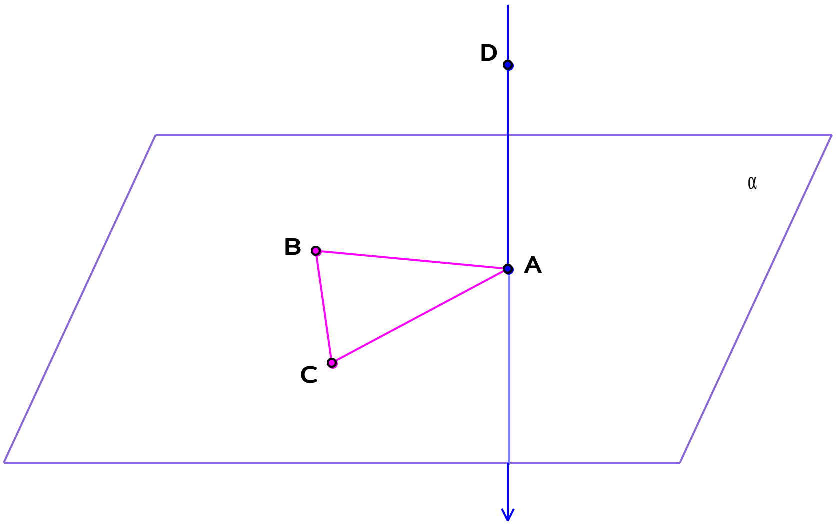

**Figure 1-1**.

## Relative Position of a Line and a Plane

A line can be in the plane, can intersect the plane in exactly one point or not intersect the plane. If a line does not intersect the plane then the **line is parralel to the plane**. In what follows we list several theorems relating lines and planes and prove only some of them.

**Theorem 1-1.** *If $b$ is a line parallel to plane $\alpha$, then any plane that contains $b$ intersect $\alpha$ in a line parallel to $\ell$.*

*Proof.* In Figure 1-2, $\beta$ is a plane that contains line $b$ and intersects $\alpha$ in the line $a$. To prove that $a \parallel b$ suppose that $a$ and $b$ intersect in a point $P$ (not shown). Since $a$ is in a plane $\alpha$, $P$ is both on line $b$ and in $\alpha$, which contradicts the hypothesis that $b$ is parallel to $\alpha$. ☐

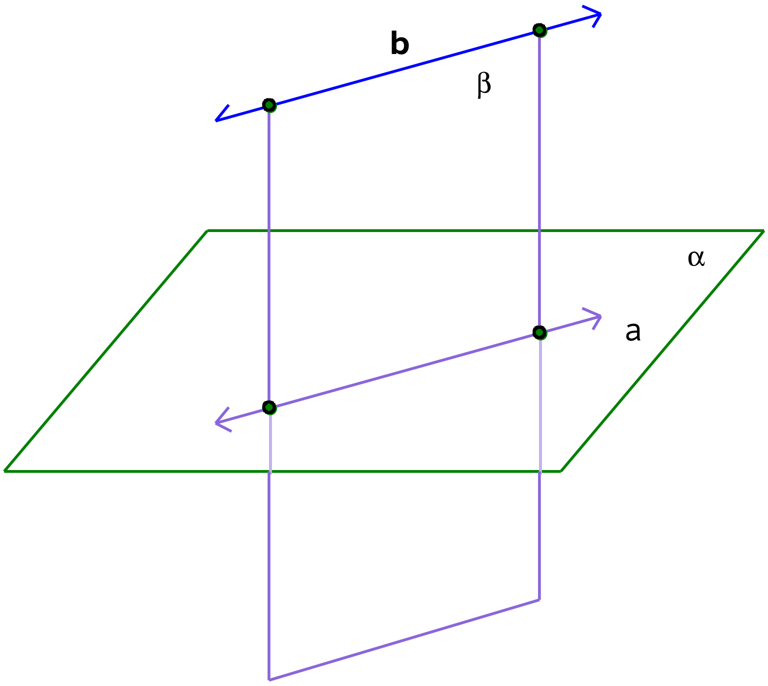

**Figure 1-2**.

An immediate consequence of Theorem 1-1 is the following:

**Corollary 1-1.** *If a line b is parallel to a plane* α *then through any point in* α *there exists a line parallel to b.*

*Proof.* Let $P$ be a point in plane α. Consider the plane β determined by $b$ and $P$. By Theorem 1-1 plane β intersect α in line $a$ parallel to $b$. Since point $P$ is in both place it is on $a$. ☐

We are now ready to prove:

**Theorem 1-2.** *A line parallel to two intersecting planes is parallel to their line of intersection.*

In Figure 1-3 line $\ell$ is parallel to planes α and β which intersect in $\overleftrightarrow{AB}$. We need to **prove that a** $\| \overleftrightarrow{AB}$.

*Proof.* We choose an arbitrary point $P$ on $\overleftrightarrow{AB}$ and consider plane π uniquelly determined by $\ell$ and $P$. Theorem 1-1 tells us that π intersects plane α in some line $a$ (not shown) parallel to $\ell$. Similarly since $\ell$ is parallel to β, it intersects β in some line $b$ (not shown). Notice that both lines $a$ and $b$ go through $P$, are in plane

$\pi$ and are parallel to $\ell$, which is also in plane $\pi$. Thus from the **Parallel Postulate** we conclude that $a = b$. Consequently line $a$ is in $\alpha$ and in $\beta$. Since the line of intersection of $\alpha$ and $\beta$ is unique it follows that $a = \overleftrightarrow{AB}$. Because $a \parallel \ell$, $\overleftrightarrow{AB} \parallel \ell$. ☐

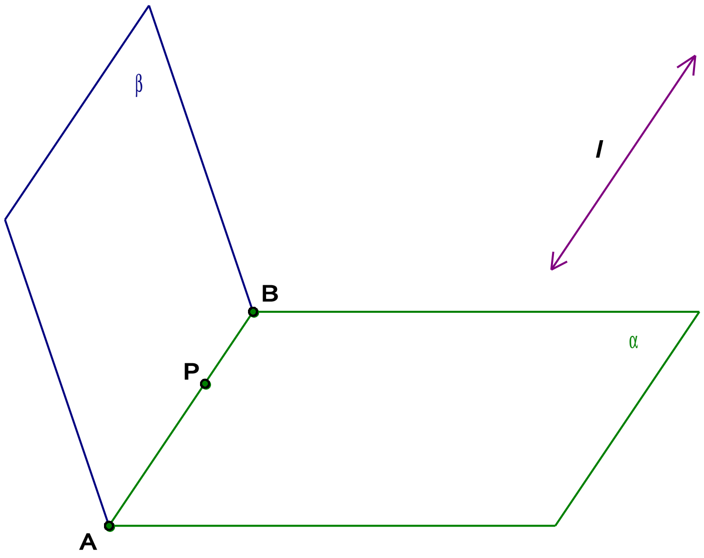

**Figure 1-3**.

We now list several theorems without proof; their proofs are straight forward.

**Theorem 1-3.** *If a, b and c are distinct lines (not necesarily in the same plane) we have:*

$$\text{if} \quad a \parallel c \quad \text{and} \quad b \parallel c \quad \text{then} \quad a \parallel c.$$

**Theorem 1-4.** *If each of two intersecting lines are parallel to plane* $\alpha$*, then the plane determined by the intersecting lines is also parallel to* $\alpha$*.*

**Theorem 1-5.** *If a plane intersects two parallel planes then two lines of intersection are also parallel.*

**NOW TRY THIS 1-2**: Draw figures illustrating Theorems 1-3, 1-4 and 1-5 and prove the theorems.

## Perpendicular and Oblique Lines

A line not in the plane and not parallel to the plane intersects the plane in a single point called the **foot of the line**. A line $n$ is **perpendicular to plane** $\alpha$ if it is perpendicular to every line in the plane, through the foot of $n$ as shown in Figure 1-4.

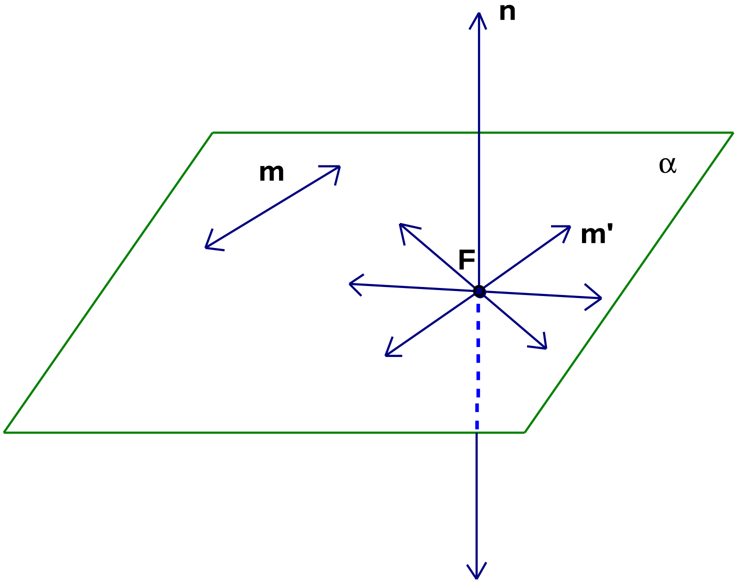

**Figure 1-4**.

If line $m$ does not go through the foot $F$, then $m$ is perpendicular to $n$ if $n$ is perpendicular to line $m'$ such that $m'$ goes through $F$ and $m'$ is parallel to $m$.

The next theorem gives a very useful criterion for a line to be perpendicular to a plane.

**Theorem 1-6.** *If a line is perpendicular to two intersecting lines then it is perpendicular to the plane determined by these lines.*

*Proof.* In Figure 1-5 line $n$ is perpendicular to two lines $p$ and $q$ in plane $\alpha$ (without loss of generality we assume that the lines goes through the foot $F$). Let $r$ be an arbitrary line through $F$ in plane $\alpha$. Given that $p \perp n$ and $q \perp n$ we need to prove that $r \perp n$. We utilize what we know about the perpendicular bisector of a segment, i.e., that a point is on the perpendicular if and only if it is equidistant from the endpoints of the segment. For that purpose choose a point $A$ on $n$ and point $D$ such that, $F$ is the midpoint of $\overline{AD}$.

Also choose points $B$ and $C$ on lines $p$ and $q$ respectively. Now $B$ and $C$ are on perpendicular bisectors of $\overline{AD}$, the first in plane $ABD$ and the second in plane $ACD$. To prove that $r \perp n$ we need only to prove that $r$ is also on the perpendicular bisector of $\overline{AD}$. For that purpose it will suffice to show that some point on $r$ is equidistant from $A$ and $D$. Let $E$ be the intersection of $\overleftrightarrow{BC}$ with $r$. We will show that $AE = DE$. This will be the case if we can prove that $\triangle ABE$ and $\triangle DBE$ are congruent. This would follow by SAS if we know that $\angle ABE$ and $\angle DBE$ are congruent. But this congruence of angles follows from the fact that $\triangle ABC \cong \triangle ACD$ (SSS). Consequently by SAS $\triangle ABE \cong \triangle DBE$ and hence $AE = ED$, which implies that $\overleftrightarrow{FE}$ is the perpendicular bisector of $\overline{AD}$. ☐

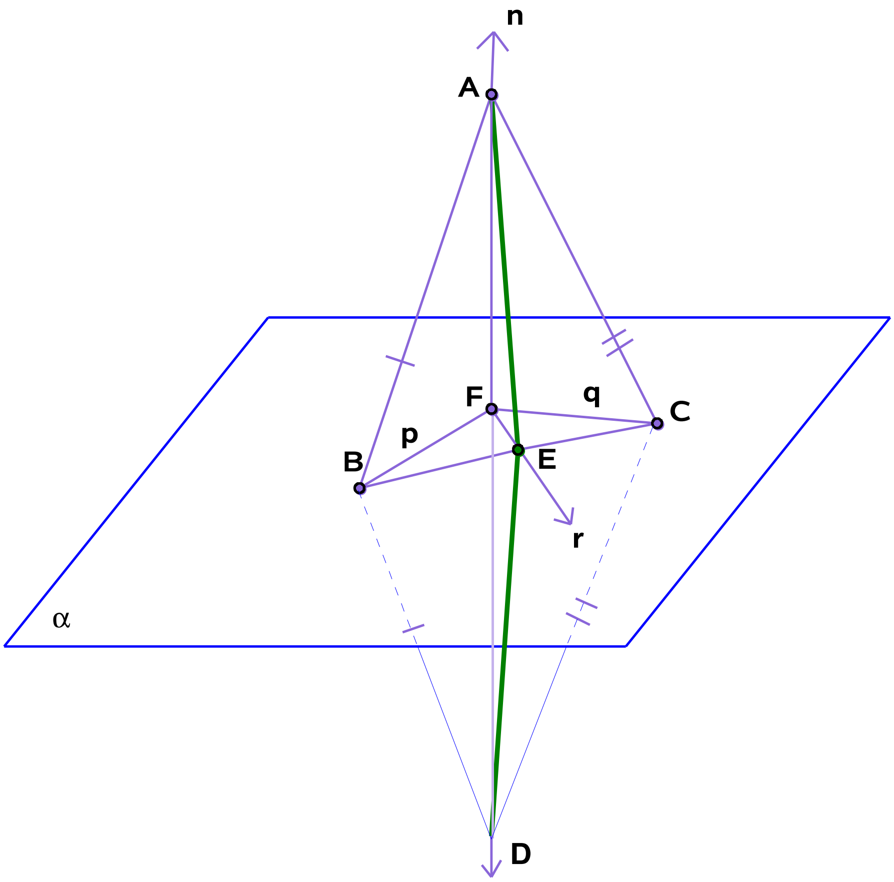

**Figure 1-5**.

The following three theorems are given without proof (the proofs are straight forward and can be found in [1]).

**Theorem 1-7.** *Two angles (not necessarily in the same plane) whose corresponding sides are parallel are either congruent or supplementary.*

**Theorem 1-8.** *If lines a and b are perpendicular to the same plane then the lines are parallel.*

**Theorem 1-9.** *Let* $\alpha$ *be a plane and a and b two lines. If* $a \perp \alpha$ *and* $b \parallel a$ *then* $b \perp \alpha$.

**NOW TRY THIS 1-3**: Write a definition of the distance from a point to a plane and prove that the length of a segment connecting a point $P$ to a point $Q$ in the plane is less or equal to the distance from $P$ to the plane.

**Definitions**

(i) A line that is neither parallel or perpendicular to a plane is a **slant line**.

(ii) The **projection** of **a point *P* onto a plane** is the foot of the perpendicular to the plane through $P$.

(iii) The **projection** of **a line or a segment onto a plane** is the set of points obtained by projecting the points on the line or the segment.

**Theorem 1-10.** *(a) The projection of a line onto a plane is a line determined by the projection of any two points on the line.*

*(b) The projection of a segment onto a plane is the segment in the plane determined by projecting the endpoints ot the segment.*

**NOW TRY THIS 1-4**: Prove part (b) of the Theorem 1-10.

The following theorem is a useful in proving many theorems in 3-D geometry.

**Theorem 1-11.** *A line in a plane perpendicular to the projection of a slant line onto the plane, is also perpendicular to the slant line.*

*Proof.* **Given**: plane $\alpha$, the slant line $\ell$, its projection $\ell'$ and line $m$ perpendicular to $\ell'$.

**Prove**: $\overleftrightarrow{BD} \perp \ell'$. □

*Proof.* In Figure 1-6, $\ell$ is a slant line with respect to plane $\alpha$. Point $A$ is on $\ell$ but not in $\alpha$. Point $B$ is the intersection of $\ell$ with $\alpha$. Without loss of generality we can assume that the line $m$ perpendicular to $\ell'$ is $\overleftrightarrow{BD}$ where $D$ is any point on the line. Consider the plane $\beta$ determined by $A$, $B$ and $A'$. Notice that because $\overleftrightarrow{AA'} \perp \alpha$ (since $A'$ is the foot of the perpendicular from $A$ to $\alpha$), $\overleftrightarrow{AA'}$ is perpendicular to every line in $\alpha$ (Theorem 1-6). Thus $\overleftrightarrow{AA'} \perp \overleftrightarrow{BD}$. Consequently $\overleftrightarrow{BD}$ is perpendicular to two lines in plane $\beta$; to $\overleftrightarrow{AA'}$ and to $\ell'$ (given). Therefore $\overleftrightarrow{BD} \perp \beta$ and by Theorem 1-6, $\overleftrightarrow{BD} \perp \ell$ (since $\ell$ is in $\beta$). ☐

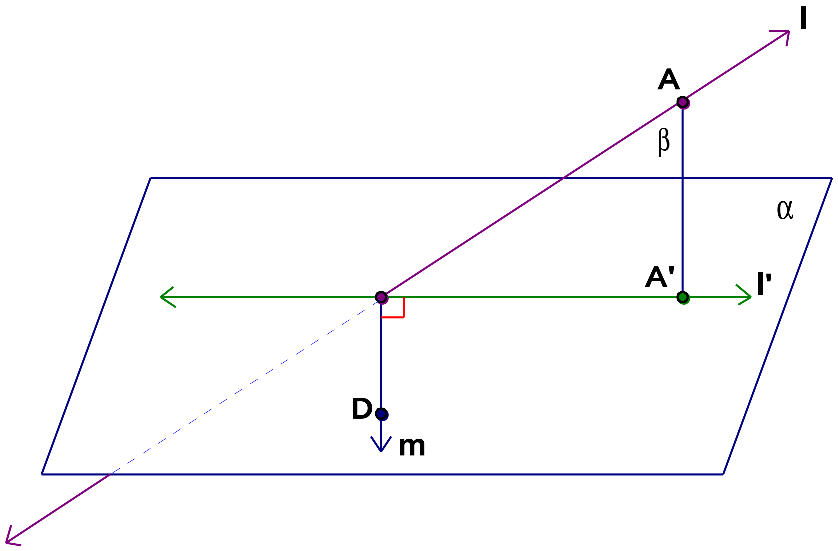

**Figure 1-6**.

**NOW TRY THIS 1-5**:

(a) State and prove a converse of Theorem 1-11.

(b) Use Theorem 1-11 to prove the following:

If $\overleftrightarrow{AB}$ is perpendicular to a plane and from $B$, the foot of the perpendicular a line $BC$ is drawn perpendicular to a line $k$ in the plane then $\overleftrightarrow{AC} \perp k$.

(c) Prove part (b) independently of Theorem 1-11 by choosing pints $D$ and $E$ on $k$ (see Figure 1-7) such that $CE = CD$. Then connect the points as shown and use properties of a perpendicular bisector.

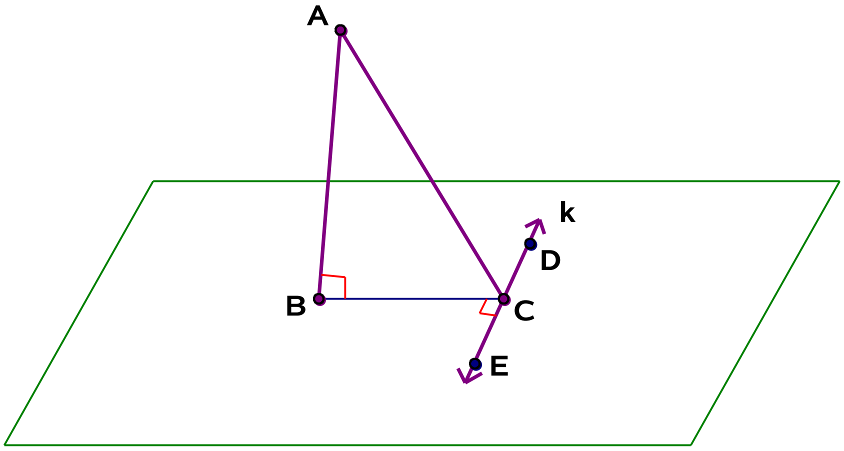

**Figure 1-7**.

**Definitions**

1. The **angle between a line and a plane** (line not parallel nor perpendicular to the plane) is the smaller of the two angles formed by the line and its projection on the plane.
2. A line in a plane divides the plane into two **half planes**. The line is called the **edge** of each half plane.
3. A **dihedral line** is formed by two intersecting planes. It is the union of two half-planes and their common edge. The half planes are the **faces**. (Notice the analogy with an angle in a plane. The sides of a dihedral angle are half planes instead of rays, and rather than meeting at a vertex, the half planes meet in a line). A dihedral angle may be thought of as generated by the rotation of a half plane about its edge.
4. Rays in the faces of a dihedral angle perpendicular to its edge at a common point on the edge form an **associated planar angle of the dihedral angle**.

   Figure 1-8 shows two associated planar angles $\angle ACB$ and $\angle A_1C_1B_1$ to the dihedral angle formed by the half planes. It can be shown that the angles are congruent (see Slaught and Lennes p. 18).
5. The **measure** of **a dihedral angle** is the measure of any of the planar associated angles.

6. Two **planes are perpendicular** if one of the dihedral angles formed by the planes is a right angle.

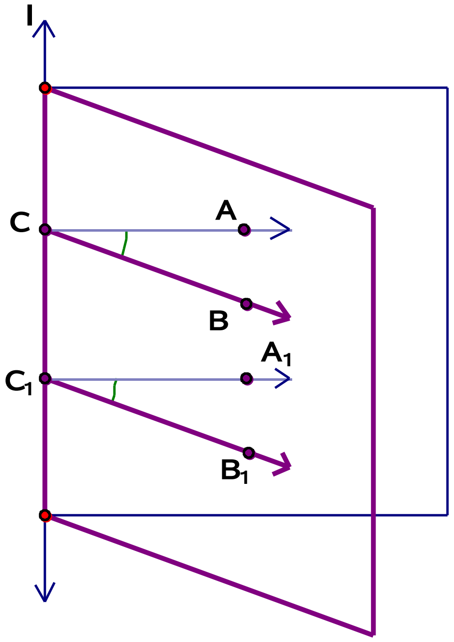

**Figure 1-8**.

**Example 1-2.** In Figure 1-9, $\ell$ is a line which intersects the plane in $B$ and makes an angle of $45^0$ with the plane. Point $A$ is on $\ell$ and the foot of the perpendicular through $A$ to the plane is $C$. Point $D$ is in the plane and $\angle ABD$ is also $45^0$. Find the measure of $\angle ABD$.

**Solution:** We drop the perpendicular from $C$ to $\overrightarrow{BD}$ intersecting $\overrightarrow{BD}$ at $E$. Notice that $\overline{EC}$ is the projection of $\overline{AE}$ on the plane. From Theorem 1-11 we conclude that because $\overleftrightarrow{BD}$ is perpendicular the projection of $\overleftrightarrow{AE}$ it is also perpendicular to $\overleftrightarrow{BE}$. Thus $\angle BEA$ is a right angle. Let $BC$ be one unit long. Then using the Pythagorean Theorem $BE = \frac{\sqrt{2}}{2}$ and $AB = \sqrt{2}$. Consequently $BE = \frac{1}{2}AB$. Therefore $m(\angle BAE) = 30^0$ and $m(\angle ABD) = 60^0$. $\square$

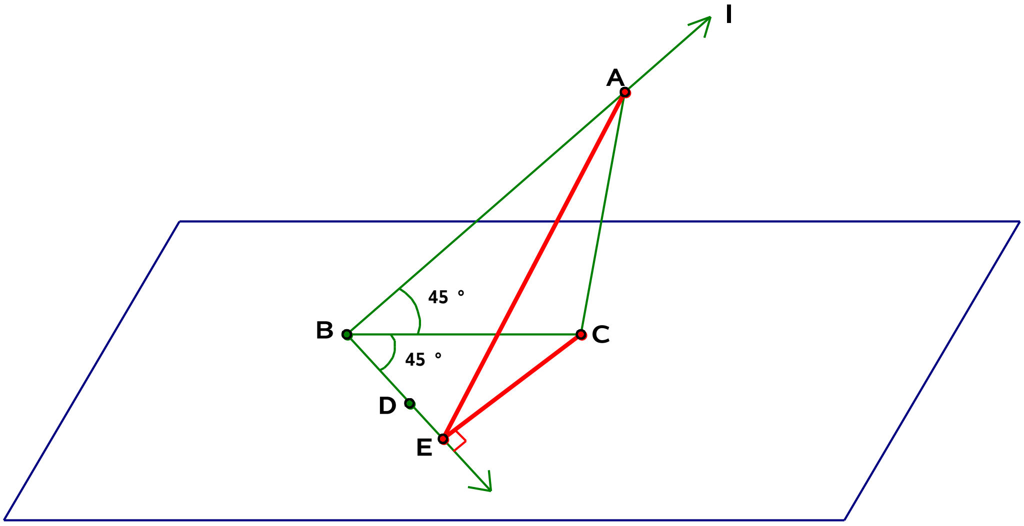

**Figure 1-9**.

Additional Topics and where to find them.

1. Slaught and Lennes *Solid Geometry*, Allyn and Bacon 1911. Also available free on the web:

   `mathcs.holycross.edu/~ahwang/pgdp/Projects/solid_geom.pdf`

   or Google: The Project Gutenberg eBook #00000: Solid Geometry

   This is an excellent treatment of solid geometry including:

   *Prism and Cylinders*, *Pyramids and Cones*, Regular and Similar Polyhedra and The Sphere.

2. Kirk Abigail, *Euler's Polyhedron Formula* at

   `http://plus.maths.org/issue43/features/kirk/index.html`

   gives a clear elementary proof of the formula.

3. **Platonic Solids**

   For a simple very elementary proof that there are only five regular polyhedra see:

   The Regular Polyhedra

   `http://members.aol.com/Polycell/regs.html`

4. For transformational geometry in 3-D as well as variety of topics in geometry including Euclidean and Non-Euclidean Geometry see the excellent book:

   Geometry by Kutuzov, B.V. (translated from Russian), SMSG University of Chicago 1960. The book is available for free download from ERIC. Google:

   ERIC ED143546 - Studies in Mathematics, Volume IV. Geometry

   Kutuzov, B.V.