

Contents

Part I	1
Preliminary Remarks	3
What Is Analysis	3
The Role of Abstraction	5
A Thought Experiment	6
0 Basic Building Blocks	9
0.1 Sets and Set Notation	9
Totally Ordered Sets	12
Collections of Sets: Indexing Sets	12
Set Operations	14
Problems	17
0.2 Functions	18
Inverse Functions	20
Images and Inverse Images	21
Problems	24
0.3 The Natural Numbers, the Integers, and Their Properties . .	27
Mathematical Induction	28
0.4 Sequences	29
Subsequences	32
Problems	34
1 The Real Numbers	37
1.1 Constructing the Axioms	37
1.2 Arithmetic	38
The "Wish List"	40
Problems	42
1.3 Order	43
Order and Arithmetic	45

vi Contents

	Absolute Values	45
	Back to the “Wish List”	46
	Problems	50
1.4	The Least Upper Bound Axiom	52
	Problems	60
2	Measuring Distances	63
2.1	Metric Spaces	63
2.2	The Euclidean Metric on \mathbb{R}^n	65
	The Cauchy–Schwarz Inequality	67
	Problems	68
3	Sets and Limits	73
3.1	Open Sets	73
	Boundedness in Metric Spaces	76
	Problems	77
3.2	Convergence of Sequences: Thinking Intuitively	81
3.3	Convergence of Sequences	82
	Problems	85
3.4	Sequences in \mathbb{R}	87
	Sequence Convergence and Order	87
	Sequence Convergence and Arithmetic	89
	Problems	91
3.5	Limit Points	93
	Problems	95
3.6	Closed Sets	96
	Problems	97
3.7	Open Sets, Closed Sets, and the Closure of a Set	98
	Problems	102
4	Continuity	105
4.1	Thinking Intuitively	105
4.2	Limit of a Function at a Point	106
	Problems	109
4.3	Continuous Functions	110
	Problems	112
4.4	Uniform Continuity	115
	Problems	116
5	Real-Valued Functions	119
5.1	Limits, Continuity, and Order	120
	Some Useful Special Cases	122

	Problems	124
5.2	One-Sided Limits	124
	Problems	126
5.3	Limits, Continuity, and Arithmetic	127
	Problems	128
6	Completeness	131
6.1	Cauchy Sequences	131
	Problems	132
6.2	Complete Metric Spaces	133
	Problems	134
7	Compactness	135
7.1	Compact Sets	136
	Problems	140
7.2	Continuity and Compactness	143
	Problems	144
7.3	Compactness in \mathbb{R}^n	145
	Problems	149
8	Connectedness	151
8.1	The Intermediate Value Theorem	151
	Problems	153
8.2	Connected Sets	153
	Problems	156
9	Differentiation of Functions of One Real Variable	159
9.1	Regarding Domains	159
9.2	The Derivative	161
	Problems	166
9.3	What Does the Derivative Tell Us about the Function? . . .	170
9.4	Proving the Mean Value Theorem	171
	Problems	174
9.5	Monotonicity and the Mean Value Theorem	176
	Problems	179
9.6	Inverse Functions	180
	Problems	182
9.7	Polynomial Approximation and Taylor’s Theorem	182
	Problems	186

viii Contents

10	Iteration and the Contraction Mapping Theorem	189
10.1	Iteration and Fixed Points	189
	Attractors and Repellers	195
	Problems	196
10.2	The Contraction Mapping Theorem	198
	Why You Should Care About Fixed Points	201
	Problems	202
10.3	More on Finding Attracting Fixed Points	205
	Problems	207
11	The Riemann Integral	209
11.1	What Is Area?	209
11.2	The Riemann Integral	210
	Problems	216
11.3	Arithmetic, Order, and the Integral	217
	Problems	218
11.4	Families of Riemann Sums	219
	The Riemann “Envelope”: Upper and Lower Sums	220
	Refinements	221
	Cauchy Criteria for the Existence of the Integral	222
	Problems	227
11.5	Existence of the Integral	228
	Problems	231
11.6	The Fundamental Theorem of Calculus	234
	Problems	235
12	Sequences of Functions	237
12.1	Pointwise Convergence	237
12.2	Uniform Convergence	240
	Problems	243
12.3	Series of Functions	246
	Problems	248
12.4	Interchange of Limit Operations	248
	Problems	254
13	Differentiating $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$	257
13.1	What Are We Studying?	257
	Problems	259
13.2	Thinking Intuitively	259
	Tangent Planes	260

13.3	Analysis in Linear Spaces	262
	Linear Transformations	265
	Linear Algebra and Analysis	267
	Problems	270
13.4	Local Linear Approximation for Functions of Several Variables	272
	Connections—Total and Partial Derivatives	274
	Problems	282
13.5	The Mean Value Theorem for Functions of Several Variables	287
	Problems	290
Part II		291
A	Truth and Provability	293
B	Number Properties	295
C	Exponents	298
C.1	Integer and Rational Powers	298
	Positive Integer Powers	298
C.2	Irrational Powers	303
D	Sequences in \mathbb{R} and \mathbb{R}^n	307
D.1	Sequence Convergence in \mathbb{R} and \mathbb{R}^n	307
D.2	Epsilonics: Playing the Game	311
	Voodoo Mathematics?	311
	Scratch Work—Devising a Strategy	312
	Problems	314
D.3	Infinite Limits	315
D.4	Some Important Special Sequences	316
E	Limits of Functions from \mathbb{R} to \mathbb{R}	319
E.1	Example Proofs	319
E.2	Epsilonics: Some General Principles	321
	Problems	322
F	Doubly Indexed Sequences	325
F.1	Double Sequences and Convergence	326
G	Subsequences and Convergence	330
G.1	Subsequential Limits	330
G.2	Limits Superior and Inferior	331

x Contents

H	Series of Real Numbers	334
H.1	Definition and Basic Properties	334
	Geometric Series	335
	Cauchy Criterion for Series Convergence	336
	N^{th} Term Test	337
	Absolute vs. Conditional Convergence	338
	Problems	340
H.2	Comparing Series	342
	Problems	343
H.3	Relatives of the Geometric Series	344
	Comparing the Root and the Ratio Tests	346
	Problems	347
H.4	Rearranging the Terms of a Series	348
	Problems	353
H.5	Multiplying Series	354
I	Probing the Definition of the Riemann Integral	358
I.1	Regular Riemann Sums	358
I.2	Why the Generality?	360
	Problems	361
J	Power Series	363
J.1	Definitions and Convergence of Power Series	364
	Problems	367
J.2	Integration and Differentiation of Power Series	368
	Problems	369
J.3	Taylor Series	370
	Problems	372
K	Everywhere Continuous, Nowhere Differentiable	374
K.1	Introduction	374
K.2	Constructing the Function	375
L	Newton’s Method	380
L.1	Setting the Stage	380
L.2	Iterating the Newton Function	383
	Problems	384
L.3	Experimenting with Newton’s Method	385
L.4	On Choosing x_0	386
L.5	Convergence Rate	387
	Problems	389

M	The Implicit Function Theorem	390
M.1	Solving Systems of Equations	390
M.2	The Implicit Function Theorem	393
	What on Earth?!?	395
	Properties of the Solution Function	398
	Problems	399
M.3	Connections: Quasi-Newton’s Methods	402
M.4	The Inverse Function Theorem	406
	Problems	407
N	Spaces of Continuous Functions	408
N.1	The Metric Space $C(K)$	410
N.2	Compactness in $C(K)$	412
N.3	The Stone–Weierstrass Theorem	414
	Problems	418
O	Solutions to Differential Equations	421
O.1	Definitions and Motivation	421
	Why Existence? Why Uniqueness?	423
O.2	Picard Iteration Route to Existence and Uniqueness	424
	Problems	426
O.3	Systems of Equations	427
	Problems	430
	Bibliography	431
	Index	433

