Applying this result to the Coulomb field in (72.2) and its potential energy function given in (72.3), it follows that for all $\mathbf{r}, \mathbf{s} \in \mathbb{R}^3 \setminus \{\mathbf{0}\}$ the work $W_{\mathbf{s},\mathbf{r}}$ required for moving the charge q against the field generated by the charge Q (at the origin) along an arbitrary path (in $\mathbb{R}^3 \setminus \{\mathbf{0}\}$) from position $\mathbf{s}$ to position $\mathbf{r}$ is given by the equation

$$W_{\mathbf{s},\mathbf{r}} = V(\mathbf{r}) - V(\mathbf{s}).$$

Since

$$V(\mathbf{s}) = \frac{qQ}{4\pi\varepsilon_0\|\mathbf{s}\|},$$

we may infer that

$$\lim_{\|\mathbf{s}\|\to\infty} V(\mathbf{s}) = 0, \tag{72.11}$$

and therefore

$$\lim_{\|\mathbf{s}\|\to\infty} W_{\mathbf{s},\mathbf{r}} = V(\mathbf{r}). \tag{72.12}$$

This result shows that the potential energy at a position $\mathbf{r}$ in the Coulomb field is equal to the work that is required for moving the charge q from an infinite distance (as expressed by the limit $\|\mathbf{s}\| \to \infty$) to the position $\mathbf{r}$! Note: a precise definition of the limits in (72.11) and (72.12) will be omitted, because our discussion was purely intuitive in its intent.

Remark. If instead of the work done against a force field we consider the work done by the field itself, then the sign in front of the path integral is positive, and Newton's second law therefore implies that

$$\begin{aligned} W &= \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt = \int_a^b m\mathbf{r}''(t) \cdot \mathbf{r}'(t)\, dt = \int_a^b \frac{m}{2}\frac{d}{dt}\|\mathbf{r}'(t)\|^2\, dt \\ &= \frac{m}{2}\|\mathbf{r}'(b)\|^2 - \frac{m}{2}\|\mathbf{r}'(a)\|^2 = K(b) - K(a). \end{aligned}$$

In words:

The work done on a moving particle by a force field $\mathbf{F}$ is equal to the difference in the particle's kinetic energy at the starting and endpoints of its path.

This observation is consistent with (72.10), because if $\mathbf{F} = -\nabla V$ is a gradient force field, then the law of the preservation of energy, as derived in Chapter 71, shows that $K(b) - K(a) = -(V(\mathbf{r}(b)) - V(\mathbf{r}(a)))$.

Additional Exercises

72.15. For each of the oriented curves shown in Figures 72.9, decide whether the path integral of the given vector field over the curve is positive, negative, or zero.

a) $\mathbf{F}(\mathbf{r}) = \dfrac{\mathbf{r}}{\|\mathbf{r}\|}$ **b)** $\mathbf{F}(\mathbf{r}) = \dfrac{\mathbf{r}}{\|\mathbf{r}\|}$ **c)** $\mathbf{F}(\mathbf{r}) = -\mathbf{r}$