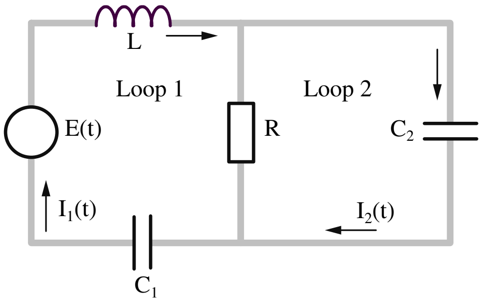

Figure 63.2: an R-C-L network of two loops.

In applying the Laplace transform, we obtain

$$\begin{aligned}\frac{3}{4}(s^2\mathcal{L}[I_1](s)-2)+\frac{4}{3}(s\mathcal{L}[I_1](s)-s\mathcal{L}[I_2](s))+\frac{5}{4}\mathcal{L}[I_1](s)&=-\frac{1}{s+1}\\ \frac{4}{3}(s\mathcal{L}[I_2](s)-s\mathcal{L}[I_1](s))+4\mathcal{L}[I_2](s)&=0,\end{aligned}$$

and the corresponding matrix form is

$$\begin{pmatrix}3s^2/4+4s/3+5/4 & -4s/3\\ -4s/3 & 4s/3+4\end{pmatrix}\begin{pmatrix}\mathcal{L}[I_1](s)\\ \mathcal{L}[I_2](s)\end{pmatrix}=\begin{pmatrix}3/2-1/(s+1)\\ 0\end{pmatrix}.$$

Multiplying both sides with the inverse of the matrix on the left yields

$$\begin{aligned}\mathcal{L}[I_1](s)&=-\frac{2(s+1)/3}{(s+1)^2+4}+\frac{8/3}{(s+1)^2+4}-\frac{2/3}{(s+1)^2}+\frac{2/3}{s+1},\\ \mathcal{L}[I_2](s)&=\frac{5(s+1)/6}{(s+1)^2+4}+\frac{10/6}{(s+1)^2+4}+\frac{1/3}{(s+1)^2}-\frac{5/6}{s+1}.\end{aligned}$$

Finally, in applying the Laplace inverse transform we find the following solutions:

$$\begin{aligned}I_1(t)&=-\frac{2}{3}e^{-t}\cos(2t)+\frac{4}{3}e^{-t}\sin(2t)-\frac{2}{3}te^{-t}+\frac{2}{3}e^{-t},\\ I_2(t)&=\frac{5}{6}e^{-t}\cos(2t)+\frac{5}{6}e^{-t}\sin(2t)+\frac{1}{3}te^{-t}-\frac{5}{6}e^{-t}.\end{aligned}$$

63.4. Exercise. Use the exponential-matrix method to verify this result.

63.5. Example. As a somewhat more complicated example, we consider the network of three loops shown in Figure 63.3. The system of differential equations is in this case

$$\begin{aligned}L_1I_1''(t)+R_1I_1'(t)+\frac{1}{C_1}(I_1(t)-I_2(t))&=E_1'(t) &&\text{(for Loop 1)},\\ L_2(I_2''(t)-I_3''(t))+R_2I_2'(t)+\frac{1}{C_1}(I_2(t)-I_1(t))+\frac{1}{C_2}I_2(t)&=0 &&\text{(for Loop 2)},\\ L_2(I_3''(t)-I_2''(t))+\frac{1}{C_3}I_3(t)&=E_2'(t) &&\text{(for Loop 3)}.\end{aligned}$$

In adding the third equation to the second, it follows that

$$\begin{aligned}L_1I_1''(t)+R_1I_1'(t)+\frac{1}{C_1}(I_1(t)-I_2(t))&=E_1'(t)\\ R_2I_2'(t)+\frac{1}{C_1}(I_2(t)-I_1(t))+\frac{1}{C_2}I_2(t)+\frac{1}{C_3}I_3(t)&=E_2'(t)\\ L_2(I_3''(t)-I_2''(t))+\frac{1}{C_3}I_3(t)&=E_2'(t).\end{aligned}$$

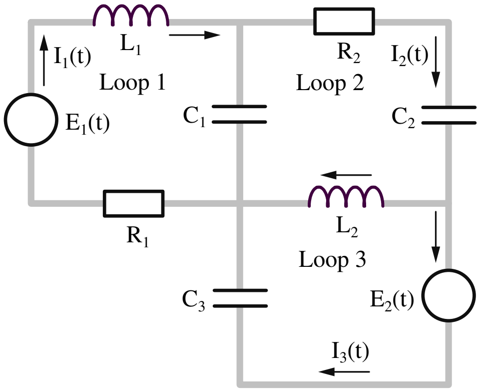

Figure 63.3: an R-C-L network of three loops.

Now introducing the substitutions

$$\begin{aligned} u_1(t) &= I_1(t), \\ u_2(t) &= I_1'(t), \\ u_3(t) &= I_2(t), \\ u_4(t) &= I_2'(t), \\ u_5(t) &= I_3(t), \\ u_6(t) &= I_3'(t), \end{aligned}$$

the equivalent first-order system is easily seen to be

$$\begin{aligned} u_1'(t) &= u_2(t) \\ u_2'(t) &= -\frac{1}{L_1C_1}u_1(t) - \frac{R_1}{L_1}u_2(t) + \frac{1}{L_1C_1}u_3(t) + \frac{1}{L_1}E_1'(t) \\ u_3'(t) &= u_4(t) \\ 0 &= \frac{1}{C_1}u_1(t) - \left(\frac{1}{C_1} + \frac{1}{C_2}\right)u_3(t) - R_2u_4(t) - \frac{1}{C_3}u_5(t) + E_2'(t) \\ u_5'(t) &= u_6(t) \\ u_6'(t) - u_4'(t) &= -\frac{1}{L_2C_3}u_5(t) + \frac{1}{L_2}E_2'(t). \end{aligned} \tag{63.3}$$

In solving the fourth of these equations for $u_4(t)$, we find that

$$u_4(t) = \frac{1}{R_2C_1}u_1(t) - \left(\frac{1}{R_2C_1} + \frac{1}{R_2C_2}\right)u_3(t) - \frac{1}{R_2C_3}u_5(t) + \frac{1}{R_2}E_2'(t),$$

and by implication,

$$\begin{aligned} u_4'(t) &= \frac{1}{R_2C_1}u_1'(t) - \left(\frac{1}{R_2C_1} + \frac{1}{R_2C_2}\right)u_3'(t) - \frac{1}{R_2C_3}u_5'(t) + \frac{1}{R_2}E_2''(t) \\ &= \frac{1}{R_2C_1}u_2(t) - \left(\frac{1}{R_2C_1} + \frac{1}{R_2C_2}\right)u_4(t) - \frac{1}{R_2C_3}u_6(t) + \frac{1}{R_2}E_2''(t) \\ &= -\left(\frac{1}{R_2^2C_1^2} + \frac{1}{R_2^2C_1C_2}\right)u_1(t) + \frac{1}{R_2C_1}u_2(t) + \left(\frac{1}{R_2C_1} + \frac{1}{R_2C_2}\right)^2 u_3(t) \\ &\quad + \left(\frac{1}{R_2^2C_1C_3} + \frac{1}{R_2^2C_2C_3}\right)u_5(t) - \frac{1}{R_2C_3}u_6(t) - \left(\frac{1}{R_2^2C_1} + \frac{1}{R_2^2C_2}\right)E_2'(t) + \frac{1}{R_2}E_2''(t). \end{aligned}$$

As we use these results to replace $u_4(t)$ and $u_4'(t)$ in (63.3), the third equation is eliminated and the total number of equations is reduced from six to five:

$$\begin{aligned}
u_1'(t) &= u_2(t)\\
u_2'(t) &= -\frac{1}{L_1C_1}u_1(t) - \frac{R_1}{L_1}u_2(t) + \frac{1}{L_1C_1}u_3(t) + \frac{1}{L_1}E_1'(t)\\
u_3'(t) &= \frac{1}{R_2C_1}u_1(t) - \left(\frac{1}{R_2C_1} + \frac{1}{R_2C_2}\right)u_3(t) - \frac{1}{R_2C_3}u_5(t) + \frac{1}{R_2}E_2'(t)\\
u_5'(t) &= u_6(t)\\
u_6'(t) &= -\left(\frac{1}{R_2^2C_1^2} + \frac{1}{R_2^2C_1C_2}\right)u_1(t) + \frac{1}{R_2C_1}u_2(t) + \left(\frac{1}{R_2C_1} + \frac{1}{R_2C_2}\right)^2 u_3(t)\\
&\quad + \left(\frac{1}{R_2^2C_1C_3} + \frac{1}{R_2^2C_2C_3} - \frac{1}{L_2C_3}\right)u_5(t) - \frac{1}{R_2C_3}u_6(t) + \frac{1}{L_2}E_2'(t)\\
&\quad - \left(\frac{1}{R_2^2C_1} + \frac{1}{R_2^2C_2}\right)E_2'(t) + \frac{1}{R_2}E_2''(t).
\end{aligned}$$

The equivalent matrix form is

$$\begin{pmatrix} u_1'(t)\\ u_2'(t)\\ u_3'(t)\\ u_5'(t)\\ u_6'(t)\end{pmatrix} = \mathbf{A}\begin{pmatrix} u_1(t)\\ u_2(t)\\ u_3(t)\\ u_5(t)\\ u_6(t)\end{pmatrix} + \begin{pmatrix} 0\\ E_1'(t)/L_1\\ E_2'(t)/R_2\\ 0\\ E_2'(t)/L_2 - (1/C_1 + 1/C_2)E_2'(t)/R_2^2 + E_2''(t)/R_2\end{pmatrix},$$

where

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0\\ -\frac{1}{L_1C_1} & -\frac{R_1}{L_1} & \frac{1}{L_1C_1} & 0 & 0\\ \frac{1}{R_2C_1} & 0 & -\frac{1}{R_2C_1} - \frac{1}{R_2C_2} & -\frac{1}{R_2C_3} & 0\\ 0 & 0 & 0 & 0 & 1\\ -\frac{1}{R_2^2C_1^2} - \frac{1}{R_2^2C_1C_2} & \frac{1}{R_2C_1} & \left(\frac{1}{R_2C_1} + \frac{1}{R_2C_2}\right)^2 & \frac{1}{R_2^2C_1C_3} + \frac{1}{R_2^2C_2C_3} - \frac{1}{L_2C_3} & -\frac{1}{R_2C_3}\end{pmatrix}.$$

63.6. Exercise. Use exponential matrices (and a computer) to solve the first-order system above for $R_1 = 500\,\Omega$, $R_2 = 100\,\Omega$, $L_1 = 0.1\,H$, $L_2 = 0.3\,H$, $C_1 = 10^{-6}\,F$, $C_2 = 2\cdot 10^{-6}\,F$, $C_3 = 4\cdot 10^{-6}\,F$, $E_1(t) = E_2(t) = \sin(1000t)$, and $I_1(0) = I_2(0) = I_3(0) = I_1'(0) = I_3'(0) = 0$. Note: as in Exercise 63.2, you may again have to work with the formulae in (61.24) or (61.25).

63.7. Example. As our final example in this chapter, we consider the R-C network in Figure 63.4, which does not involve any inductors, but only capacitors and resistors. As in the preceding examples,

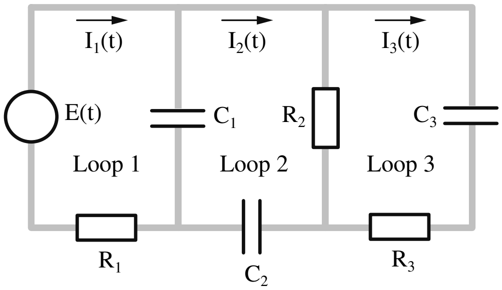

Figure 63.4: an R-C network of three loops.