is invertible, and the values of its inverse function are denoted by $\cosh^{-1}(x)$ *or* $\operatorname{arcosh}(x)$ *(hyperbolic area cosine). The values of* $\operatorname{arcosh}(x)$ *are given by the equation*

$$\operatorname{arcosh}(x) = \ln\left(x + \sqrt{x^2 - 1}\right) \text{ for all } x \in [1, \infty),$$

and the domain and range of the hyperbolic area cosine are $[1, \infty)$ *and* $[0, \infty)$ *respectively.*

★ *Remark.* The use of the prefix "area" in denoting the hyperbolic inverse functions is motivated by the following geometric fact: let (x, y) be a point in the first quadrant (i.e., $x, y \geq 0$) on the hyperbola given by the equation $x^2 - y^2 = 1$ (see Figure 31.3), and let $t \in \mathbb{R}$ such that $(x, y) = (\cosh(t), \sinh(t))$ or equivalently $t = \operatorname{arcosh}(x) = \operatorname{arsinh}(y)$ (see the ⋆remark⋆ on p.237). Then the difference of the *areas* A and B shown in Figure 31.3 on the left, is equal to t (see Exercise 31.7). Furthermore, the area of the sector C shown on the right is equal to $t/2$ because $C = A - (A + B)/2 = (A - B)/2$.

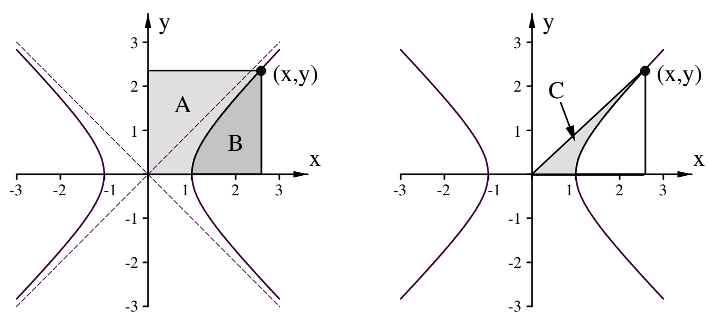

Figure 31.3: area property of the hyperbolic inverse functions.

31.7. Exercise. Let $t \in [0, \infty)$ and $(x, y) := (\cosh(t), \sinh(t))$. Prove that the areas A and B, shown in Figure 31.3, satisfy the equation $A - B = t$. *Hint.* Use integration to express the difference $A - B$ as a function of x, and prove that $A - B - t$ is constant by demonstrating that the derivative of $A - B - t$ with respect to x is zero. Then set x equal to 1 to show that $A - B - t = 0$. ★

Inverse Integration by Substitution

31.8. Example. We wish to find the value of the definite integral

$$\int_0^1 \frac{1}{\sqrt{1 + x^2}}\, dx.$$

With $x := \sinh(t)$ and $dx = \cosh(t)\, dt$ (see (31.8)) the rule of integration by substitution, as stated in Theorem 30.1, implies that

$$\int_{\operatorname{arsinh}(0)}^{\operatorname{arsinh}(1)} \frac{\cosh(t)}{\sqrt{1 + \sinh^2(t)}}\, dt = \int_0^1 \frac{1}{\sqrt{1 + x^2}}\, dx.$$

(Note: in order to find the boundaries for the integral on the right, we used the fact that $\sinh(\operatorname{arsinh}(x)) = x$ for all $x \in \mathbb{R}$.) The trick is now to eliminate the root in the denominator of the integrand on the left-hand side by using the hyperbolic theorem of Pythagoras in (31.7). This yields

$$\int_0^1 \frac{1}{\sqrt{1 + x^2}}\, dx = \int_{\operatorname{arsinh}(0)}^{\operatorname{arsinh}(1)} \frac{\cosh(t)}{\sqrt{\cosh^2(t)}}\, dt = \int_{\operatorname{arsinh}(0)}^{\operatorname{arsinh}(1)} \frac{\cosh(t)}{|\cosh(t)|}\, dt.$$